

SIMULTANEOUS INFERENCE IN
POSSIBLY INCOMPLETE
MULTIVARIATE REGRESSIONS WITH
APPLICATION TO ASSET PRICING

MARIE-CLAUDE BEAULIEU
JEAN-MARIE DUFOUR
LYNDA KHALAF
OLENA MELIN

Les **cahiers de la série scientifique** visent à rendre accessibles les résultats des recherches effectuées par des chercheurs membres du CIRANO afin de susciter échanges et commentaires. Ces cahiers sont rédigés dans le style des publications scientifiques et n'engagent que leurs auteurs.

The purpose of the Working Papers is to disseminate the results of research conducted by CIRANO research members in order to solicit exchanges and comments. These reports are written in the style of scientific publications. The ideas and opinions expressed in these documents are solely those of the authors.

Le **CIRANO** est un organisme sans but lucratif constitué en vertu de la Loi des compagnies du Québec. Le financement de son infrastructure et de ses activités de recherche provient des cotisations de ses organisations-membres, d'une subvention d'infrastructure du gouvernement du Québec, de même que des subventions et mandats obtenus par ses équipes de recherche.

CIRANO is a private non-profit organization incorporated under the Quebec Companies Act. Its infrastructure and research activities are funded through fees paid by member organizations, an infrastructure grant from the government of Quebec, and grants and research mandates obtained by its research teams.

Les partenaires du CIRANO – CIRANO Partners

Partenaires Corporatifs - Corporate Partners

Autorité des marchés financiers
Banque de développement du Canada
Banque du Canada
Banque Nationale du Canada
Bell Canada
BMO Groupe financier
La Caisse
Énergir
Hydro-Québec
Intact Corporation Financière
Mouvement Desjardins
Power Corporation du Canada
Pratt & Whitney Canada
VIA Rail Canada

Partenaires gouvernementaux - Governmental partners

Ministère des Finances du Québec
Ministère de l'Économie, de l'Innovation et de l'Énergie
Innovation, Sciences et Développement Économique
Canada
Ville de Montréal

Partenaires universitaires - University Partners

École de technologie supérieure
École nationale d'administration publique
HEC Montréal
Institut national de la recherche scientifique
Polytechnique Montréal
Université Concordia
Université de Montréal
Université de Sherbrooke
Université du Québec
Université du Québec à Montréal
Université Laval
Université McGill

Le CIRANO collabore avec de nombreux centres et chaires de recherche universitaires dont on peut consulter la liste sur son site web. CIRANO collaborates with many centres and university research chairs; list available on its website.

© Octobre 2026. Marie-Claude Beaulieu, Jean-Marie Dufour, Lynda Khalaf, Olena Melin. Tous droits réservés. *All rights reserved.* Reproduction partielle permise avec citation du document source, incluant la notice ©. *Short sections may be quoted without explicit permission, if full credit, including © notice, is given to the source.*

Les idées et les opinions émises dans cette publication sont sous l'unique responsabilité des auteurs et ne représentent pas les positions du CIRANO ou de ses partenaires. *The observations and viewpoints expressed in this publication are the sole responsibility of the authors; they do not represent the positions of CIRANO or its partners.*

Simultaneous inference in possibly incomplete multivariate regressions with application to asset pricing^{*}

Marie-Claude Beaulieu[†], Jean-Marie Dufour[‡], Lynda Khalaf[§], Olena Melin^{**}

Abstract

This paper presents finite-sample tests and confidence procedures for a multivariate linear regression system with cross-dependent possibly non-Gaussian disturbances, allowing for potentially missing explanatory variables. We consider a wide class of multivariate statistics based on the roots of a determinantal equation involving restricted and unrestricted least squares residuals. We characterize the exact distribution of these roots and identify important pivotal cases. General conditions for asymptotic validity are also given. To deal with missing explanatory variables, we introduce an identifiable ex-post parameter which embeds the impact of left-out variables, and show that usual tests remain valid from a conservative perspective. We derive simultaneous confidence bands for a vector (as well as linear combinations) of regression coefficients. The results are applied to asset pricing, with focus on risk-return analysis. We revisit the distinction between the alpha (risk-adjusted outperformance) and beta (risk component) of returns, from our ex-post parametrization perspective. Empirically, we analyze the risk-return profile of Catastrophe bond mutual funds.

Résumé

Cet article présente des tests et des procédures de construction d'intervalles de confiance en échantillon fini pour un système de régressions linéaires multivariées dont les termes d'erreur peuvent présenter des dépendances croisées et ne pas suivre une distribution gaussienne, tout en permettant l'omission éventuelle de variables explicatives. Nous considérons une vaste

^{*} This work was supported by the Social Sciences and Humanities Research Council of Canada, the Natural Sciences and Engineering Research Council of Canada, the Chaire RBC en innovations financières (Université Laval), and the William Dow Chair of Political Economy (McGill University).

[†] Centre de recherche sur les risques, les enjeux économiques et les politiques publiques (CRREP), Département de finance et assurance, Pavillon Palasis-Prince, local 3630, Université Laval, Québec, Canada G1K 7P4. FAX: (418) 656-2624. email: Marie-Claude.Beaulieu@fas.ulaval.ca

[‡] William Dow Professor of Economics, McGill University; Centre interuniversitaire de recherche en économie quantitative (CIREQ); Centre interuniversitaire de recherche en analyse des organisations (CIRANO). Mailing address: Department of Economics, Leacock Building, Rm 519, 855 Sherbrooke Street West, Montréal, Québec, Canada H3A 2T7. TEL: (438) 349 4738; FAX: (514) 398 4938. email: jean-marie.dufour@mcgill.ca.

[§] Economics Department, and Centre for Monetary and Financial Economics (co-director), Carleton University. Mailing address: Economics Department, Carleton University, Loeb Building 1125 Colonel By Drive, Ottawa, Ontario, K1S 5B6 Canada. Tel (613) 520-2600-8697; FAX: (613)-520-3906. email: Lynda.Khalaf@carleton.ca.

^{**} Bank of Canada, 234 Wellington street, Ottawa, Ontario, K1A 0G9, Canada. e-mail: OMelin@bank-banque-canada.ca. The views expressed in this paper do not represent those of the Bank of Canada.

classe de statistiques multivariées fondées sur les racines d'une équation déterminantielle faisant intervenir les résidus des moindres carrés des modèles restreints et non restreints. Nous caractérisons la distribution exacte de ces racines et identifions d'importants cas où les statistiques sont pivots. Nous établissons également des conditions générales de validité asymptotique. Pour traiter le problème des variables explicatives omises, nous introduisons un paramètre ex post identifiable qui intègre l'effet de ces variables et montrons que les tests usuels demeurent valides dans une perspective conservatrice. Nous établissons des bandes de confiance simultanées pour un vecteur de coefficients de régression, ainsi que pour leurs combinaisons linéaires. Les résultats sont appliqués à l'évaluation des actifs financiers, en mettant l'accent sur l'analyse du risque et du rendement. Nous réexaminons la distinction entre l'alpha (surperformance ajustée au risque) et le bêta (composante du risque) des rendements, à la lumière de notre paramétrisation ex post.

Sur le plan empirique, nous analysons le profil risque-rendement des fonds communs de placement investissant dans des obligations catastrophes.

Mots-clés/Keywords : Régression multivariée, hypothèse linéaire mixte uniforme, test exact, régression incomplète, décomposition risque-rendement, inférence simultanée, profil risque-rendement / Multivariate regression, uniform mixed linear hypothesis, exact test, incomplete regression, risk-return decomposition, simultaneous inference, risk-return profiling.

Codes JEL/JEL Codes : C12, C30, C58

Pour citer ce document / To quote this document

Beaulieu, M.-C., Dufour, J.-M., Khalaf, L., & Melin, O. (2026). Simultaneous inference in possibly incomplete multivariate regressions with application to asset pricing (2026s-22, Cahiers scientifiques, CIRANO.) <https://doi.org/10.54932/QFBE7492>

Contents

1	Introduction	1
2	Framework and test statistics	5
2.1	Tested hypotheses and special cases	6
2.2	Estimators and test criteria	7
3	Distributions of test criteria: baseline regression	9
4	Incomplete multivariate regression	10
4.1	Missing regressors in a multivariate linear regression	10
4.2	Incomplete linear asset pricing model	12
5	Asymptotic validity	15
6	Simultaneous inference on factor loadings	16
7	Empirical analysis	18
8	Conclusion	25
A	Proofs	A–1
B	Data and Mimicking Portfolios	A–5

1 Introduction

Inference on regression coefficients is a fundamental problem in multivariate analysis, with a broad range of applications; see Rao (1973, Chapter 8), Anderson (1984, Chapter 8), Berndt and Savin (1977), and Dufour and Khalaf (2002). The focus of this paper is on tests and confidence procedures (using primarily a finite-sample approach) based on multivariate least-squares (**LS**), for a system of n equations (where n can be large) with cross-dependent possibly non-Gaussian disturbances, and allowing for missing explanatory variables. We study tests of *uniform mixed* linear (**UML**) hypotheses on regression coefficients, and confidence sets on individual coefficients. The contribution of an explanatory variable is often interpreted system-wide; we thus aim for a simultaneous finite-sample methodology which is well-suited for such tasks.¹ Empirically, we analyze the joint drivers of risk and returns for n financial assets, allowing for possibly missing risk factors.²

A finite-sample perspective is important in many applied contexts, including asset pricing. While estimators and classical test statistics have standard limiting distributions, underlying approximations are known to perform poorly unless n is small. Distortions typically stem from three related problems. The first source involves nuisance parameters; for example, the number of free parameters in the error covariance matrix is 55 for $n = 10$, and 325 with $n = 25$. This poses concrete limitations, in particular for time series observed at low (say monthly or quarterly) frequencies, or when temporal constancy restricts attention to short time spans. Heavy tails also constitute a common problem in financial data, which motivates explicitly allowing for non-Gaussian disturbances.

The UML specification [Berndt and Savin (1977)] means that the same linear restrictions – hence the *uniform* descriptor – affect the corresponding coefficients of all equations – hence the *mixed* descriptor – possibly after a linear transformation of the equations. This feature is common in multivariate linear asset pricing models. Such joint restrictions yield *component* uniform hypotheses of lower dimension $g \leq n$ [Olkin and Tomskey (1981)]. In finite samples, the properties of combined relationships are non-trivial beyond the Gaus-

¹For general references on simultaneous inference, see Scheffé (1959), Savin (1984), Dufour (1989), Abdelkhalek and Dufour (1998), Jordà (2009), Jordà and Marcellino (2010), Bolduc, Khalaf and Yélou (2010), and Montiel Olea and Plagborg-Møller (2019).

²For finite-sample applications in line with our focus, see Beaulieu, Dufour and Khalaf (2007, 2010, 2013, 2025), Kleibergen, Lingwei and Zhan (2023), Beaulieu, Dufour, Khalaf and Melin (2023), and Kleibergen et al. (2023).

sian case, an issue which motivates our work. Combinations play an important role in many fields. In asset pricing where evidence against normality abounds, combinations correspond to portfolio formation.

Many frameworks have been proposed in statistics to model non-normality, featuring for example elliptical, sub-Gaussian and mixtures distributions. Alternatively, a familiar view is that non-normality reflects left-out factors. In this case, the regression disturbance may be viewed as the sum of two terms: a zero-mean Gaussian noise and a random component which captures asymmetries, fat tails, or some unobserved explanatory variable. It is natural to question model completeness assumptions, so the missing variables scenario is of broad interest. However, not much work is available on finite-sample inference in incomplete regressions; exceptions include Beaulieu et al. (2023) and Beaulieu, Dufour and Khalaf (2025), but these contributions are restricted to Hotelling-type tests.

Motivated by the above, this paper makes four contributions. We work in the context of fixed equation number n and sample size T , where $T > n$. We consider standard statistics which depend on the data through the roots of a determinantal equation involving the restricted and unrestricted LS residual sum of squares. A broad spectrum of test statistics - including LR and Wald criteria - can be derived through the class of elementary symmetric functions of these roots [Rao (1973, p. 74), Olkin and Tomskey (1981)].

Our main contributions, can be summarized as follows. *First*, we characterize the finite-sample distribution of these roots. Specifically, we derive the underlying intervening parameters in a general setting and then define concrete structures where nuisance parameters can be evacuated in finite samples. These include: (i) assumptions on the form of the restrictions tested; (ii) restrictions on the error covariance, and (iii) conditions on the error distributions. In particular, we generalize the invariance results of Zellner (1976) to multivariate regressions. Furthermore, the special case considered by Moreira and Moreira (2019) leading to a Kronecker product variance/covariance form is also covered; see also Guggenberger, Kleibergen and Mavroeidis (2024), Kleibergen and Mavroeidis (2023).

Our *second* contribution is the *missing regressor* scenario. In this case, we define an identifiable *ex-post* (*i.e.* data-dependent) parameter which embeds the impact of the left-out components [for a related perspective, see Berk, Brown, Buja, Zhang and Zhao (2013)]. The latter is a pseudo-true value which collapses into the original regression coefficient when the included and excluded variables are orthogonal. We study the usual UML tests

which are applied to the observed data. Building on the Courant-Fischer theorem of linear algebra [see *e.g.* Horn and Johnson (1990, Theorem 4.2.11)], we show that these tests are level-correct (in finite samples) for inference on the reparametrized inference target, in the sense that rejections are compelling. Pseudo-true values may not fully reflect the economic content of the original parameters; for a recent perspective, see Hansen and Lee (2021) or Andrews and Kwon (2024). Nevertheless, the literature is moving towards incorporating misspecification into the economic foundations of underlying models, thus providing useful interpretations of pseudo-true values including ex-post formulations.³

Third, we exploit our finite-sample theory to derive exact simultaneous confidence bands for a vector of regression coefficients, or for a combination thereof. When the regression is incomplete, these bands cover the ex-post pseudo-coefficient with probability at least as large as the nominal level of the procedure. Concretely, we invert the Hotelling-type statistics which constitute a special case of the UML hypotheses criteria which we have analyzed. This entails formulating a joint confidence region for the vector in question, then projecting this region onto each dimension. We derive analytical solutions for this problem, in Gaussian and non-Gaussian contexts, where cut-off points are obtained by simulation in the latter case. Our method of proof allows to show the following: (i) the joint confidence region cannot be empty; (ii) projections are bounded; and (iii) the form of the resulting bands is similar to its single-equation counterpart, up to a critical point correction. Inference is thus possible through individual regressions.

Fourth, we show that the proposed exact inference procedures remain asymptotically under standard large-sample assumptions. As the result, a result, using the finite-sample distributional theory does involve a cost in terms of reliability in large samples.

Our *fifth* contribution pertains to asset pricing, with focus on risk-return analysis given a set of returns on financial assets. Risk-return analysis involves decomposing returns into the so-called *alpha* and *beta* fundamentals. Broadly defined, *beta* is the portion of returns which can be attributed to risk factors, while *alpha* represents the component which cannot be explained by any exposure to such factors. Concretely, the above components are estimated as the vector of intercepts (*alpha*) and the set of slope vectors (*betas*) in a multivariate regression of the considered returns on a constant and a given set of returns

³Examples in asset pricing that are relevant to our empirical framework include Giglio and Xiu (2021), Antoine, Renault and Proulx (2020), Gospodinov, Kan and Robotti (2014), Kan, Robotti and Shanken (2013), Beaulieu et al. (2023) and Beaulieu et al. (2025).

on observed risk factors.

A strict interpretation of regression parameters as fundamental *alpha* and *betas* rests on the assumption of complete risk-adjustment, in the sense that no hidden source of risk differential was ignored. This assumption seems to lack support, given the prolific literature on explaining returns. Where an apparent outperformance (a positive *alpha*) emerges, it is thus likely that a hidden risk factor has yet to be identified; see Ardia, Barras, Gagliardini and Scaillet (2024) for a recent perspective. Related skepticism about standard return profiling has a long tradition in asset pricing, as expressed for example by Cochrane (2007): “*there is no alpha and beta, there is just beta you know about, and beta you don’t understand yet, and no clear separation between the two*”. Another insightful related discourse on “*alphas becoming betas*”, that may be traced back to the model of Lo (2004), refers to the consequences or manifestations of strategies seeking or finding *alpha*, on the cross-section of risk. For example, Back, Crane and Crotty (2018) and Schneider and Zechner (2020) document a significant trade-off, through either low-risk or inefficiency seeking strategies, between a coskewness factor and CAPM *alpha*; Cho (2020) uncovers an endogenous relation between arbitrage capital factors and the pursuit of CAPM *alpha*. In sum, a parameter which integrates the effects of an unobserved risk factor provides a more concrete basis for risk-return decompositions. We show that associated rejections remain compelling despite missing factors.

Empirically, we analyze the risk-return profile of Catastrophe [Cat] bond mutual funds [CBMFs]. Launched nearly two decades ago, CBFMs offer an investment in a diversified portfolio of Cat bonds which pay coupons and principal if no default-triggering catastrophe (*e.g.* tornado, earthquake) falls out. Interest in these assets has grown substantially; nearly US \$36.5 billion in Cat bonds were outstanding in 2022 with Insurance Linked Security fund managers absorbing almost three-quarters of the outstanding volume [Braun and Kousky (2021)]. We find insignificant exposure to common sources of risk including coskewness, with Gaussian and fat-tailed distributions. However, following the tsunami that struck Japan in 2011, confidence bands for *alpha* across considered models are positive for most funds. In contrast, the effects of several hurricanes in US are muted. These findings support outperformance, despite the possibility of missing factors. Results further contribute to the behavioral literature (reviewed below) which documents disparities in the effects of unfamiliar versus foreseeable stress scenarios, even if these are unlikely.

The paper is organized as follows. Section 2 defines the considered model, inference problem and classical test statistics. Our key distributional results are presented in Section 3. In Section 6, we derive simultaneous bands which invert Hotelling-type statistics. Our empirical analysis is reported in Section 7 and Section 8 concludes. An appendix provides proofs of the theorems and further details on data.

2 Framework and test statistics

Multivariate linear regressions take the general form:

$$Y = XB + U, \quad U = WJ' \Leftrightarrow Y_t = B'X_t + U_t, \quad U_t = JW_t, \quad t = 1, \dots, T, \quad (2.1)$$

where: $Y = [Y_{it}]$ is a $T \times n$ matrix of observations on n dependent variables Y_{it} ($i = 1, \dots, n$); X is a $T \times k$ full-column rank matrix of exogenous variables; Y'_t and X'_t are the t -th rows of Y and X respectively; B is an unknown $k \times n$ matrix; J is an unknown nonsingular (possibly random) $n \times n$ matrix; U'_t is the t -th row of U ; W is a $T \times n$ matrix of random errors, and W'_t is the t -th row of W . The joint distribution of $W_1, \dots, W_T$ [or the distribution of $\text{vec}(W)$] is either fully specified, or specified up to a nuisance parameter κ , where $\text{vec}(\cdot)$ refers to the operator which stacks the columns of a matrix into a vector. For finite-sample statistical analysis, it will be assumed that we can condition on X . Multivariate regressions of the form (2.1) are broadly applied in statistics and econometrics. Model (2.1) is taken here as a representation of the conditional distribution of Y given X , so X can be taken as fixed. Each column of B is the regression coefficient vector for a given dependent variable, while each row of B represent the coefficients on a given explanatory variable in different regression equations. If X is random, this has the advantage of eliminating the need to specify the distribution of X . For inference, we will need to spell out assumptions on the error matrix U .

Throughout the paper, we use the following notations. ι_m refers to an $m \times 1$ vector of ones; I_m denotes the $m \times m$ identity matrix. $s_m[j]$ denotes an $m \times 1$ selection vector with all elements equal to zero except for the j -th element which equals 1. $\mathcal{P}[A] = A(A'A)^{-1}A'$ and $\mathcal{M}[A] = I_m - \mathcal{P}[A]$ for any full column rank $m \times r$ matrix A . and We refer to a $1 - \alpha$ level confidence set for a parameter as $\text{CS}_\alpha(\cdot)$. $F(m_1, m_2)$ denotes the Fisher distribution with degrees of freedom m_1 and m_2 , and $\chi^2(m)$ indicates the χ^2 distribution with degrees of freedom m .

2.1 Tested hypotheses and special cases

We study tests of UML hypotheses which take the form:

$$\tilde{H} : CBG = D \text{ for known } C, G \text{ and } D \quad (2.2)$$

where C is $c \times k$ with rank c , $0 \leq c \leq k$, and G is $n \times g$ with rank g , $1 \leq g \leq n$. For general discussion of the UML framework, see Berndt and Savin (1977), Dufour and Khalaf (2002) and references therein. Financial applications are discussed in Beaulieu, Dufour and Khalaf (2007, 2010, 2013, 2025). For future reference, we set:

$$n := \min(c, g), \quad \gamma := [(\text{vec}(C))', (\text{vec}(D))']'. \quad (2.3)$$

$\tilde{H}$ can be rewritten in terms of the transformed regression of YG on X , where YG combines the n original endogenous variables linearly into a lower dimension g . In finance, such transformations correspond to portfolio formation.

For some inference problems, we shall consider the following matrix mixture model:

$$W = V\mathcal{Z}(n) \quad (2.4)$$

where V is a $T \times T$ full-rank random matrix, and $\mathcal{Z}(n)$ is a $T \times n$ matrix of standard normal variables independent of V : if we denote the t -th row of $\mathcal{Z}(n)$ by $\mathcal{Z}_t(n)'$, then

$$\mathcal{Z}_t(n) \stackrel{i.i.d.}{\sim} N[0, I_n] \quad (2.5)$$

or the matrix-normal assumption [see Dawid (1981)] holds, *i.e.* $\text{vec}[\mathcal{Z}(n)] \stackrel{i.i.d.}{\sim} N[0, I_{nT}]$.

Assumption (2.4) includes various n -dimensional elliptically contoured distributions and skew-elliptical distributions. Special cases of (2.4) include the normal distribution with $V = I_T$, and the multivariate Student-t distribution with κ degrees-of-freedom, where V is a diagonal matrix and diagonal terms denoted V_{tt} , $t = 1, \dots, T$, take the form:

$$V_{tt} = 1/(v_t/\kappa)^{1/2}, \quad v_t \stackrel{i.i.d.}{\sim} \chi^2(\kappa) \text{ independently of } \mathcal{Z}(n). \quad (2.6)$$

If $V_{tt} = (v_t)^{1/2}$ where v_t are *i.i.d.* positive variables that follow a stable distribution with location zero, scale 1, stability index $\alpha/2$ and skewness coefficient $[\cos(\pi\alpha/4)]^{2/\alpha}$, then W_t is a sub-Gaussian α -stable vector. When both V and J are non-random, $(X'X)^{-1/2}X'V\mathcal{Z}(n)J'$ is normally distributed with variance $JJ' \otimes (X'X)^{-1/2}X'VX(X'X)^{-1/2}$ which corresponds to the Kronecker structure considered by Moreira and Moreira (2019), ignoring their normalization; see also Guggenberger et al. (2024) and Kleibergen and Mavroeidis (2023). Empirically, we focus on the random case through (2.6).

Another special case is also considered for its relevance in economics, namely:

$$G = \begin{bmatrix} I_g \\ 0 \end{bmatrix}, \quad J = \begin{bmatrix} J_1 & 0 \\ J_2 & J_3 \end{bmatrix} \quad (2.7)$$

where J_1 and J_3 are invertible $g \times g$ and $(n-g) \times (n-g)$ matrices, so G is a selection matrix, and J is a conformable lower block diagonal matrix. This set-up represents (uniform) linear parameter restrictions on a subset of g equations, along with an (ordered) causal structure between the corresponding disturbances (in the sense that UG causes the remaining block of disturbances).

2.2 Estimators and test criteria

The LS estimators under the restrictions (2.2) and the associated test statistics have tractable non-iterative forms [Berndt and Savin (1977), Gouriéroux, Monfort and Renault (1995), Dufour and Khalaf (2002)]. The restricted estimators are:

$$\tilde{B}(\gamma, G) = \hat{B} - (X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}(C\hat{B}G - D)(G'\hat{S}G)^{-1}G'\hat{S}, \quad (2.8)$$

$$\tilde{S}(\gamma, G) = \hat{S} + \hat{S}G(G'\hat{S}G)^{-1}(C\hat{B}G - D)'[C(X'X)^{-1}C']^{-1}(C\hat{B}G - D)(G'\hat{S}G)^{-1}G'\hat{S}, \quad (2.9)$$

where $\hat{B} = (X'X)^{-1}X'Y$, $\hat{S} = \hat{U}'\hat{U}$, $\hat{U} = Y - X\hat{B}$, $\tilde{S}(\gamma) = \tilde{U}(\gamma, G)'\tilde{U}(\gamma, G)$, and $\tilde{U}(\gamma, G) = Y - X\tilde{B}(\gamma, G)$. Standard test statistics depend on $\mu_1(\gamma, G) \geq \dots \geq \mu_n(\gamma, G)$, the non-zero roots of the determinantal equation

$$\det([S(\gamma, G) - G'\hat{S}G] - \mu G'\hat{S}G) = 0, \quad (2.10)$$

$$S(\gamma, G) := G'\hat{S}G + (C\hat{B}G - D)'[C(X'X)^{-1}C']^{-1}(C\hat{B}G - D). \quad (2.11)$$

Specifically, the LR and Wald criteria $[\mathcal{LR}(\gamma, G)$ and $\mathcal{T}(\gamma, G)$] can be expressed as:

$$\mathcal{LR}(\gamma, G) = -T \ln[\mathcal{L}(\gamma, G)], \quad \mathcal{L}(\gamma, G) = \frac{|\hat{S}|}{|\tilde{S}(\gamma, G)|} = \prod_{i=1}^n [1 + \mu_i(\gamma, G)]^{-1}, \quad (2.12)$$

$$\mathcal{T}(\gamma, G) = T \operatorname{tr}\{\hat{S}^{-1}[\tilde{S}(\gamma, G) - \hat{S}]\} = T \sum_{i=1}^n \mu_i(\gamma, G). \quad (2.13)$$

Our notation underscores the dependence on G . When $G = I_n$, we simply write:

$$\tilde{B}(\gamma) := \tilde{B}(\gamma, I_n), \quad \tilde{U}(\gamma) := \tilde{U}(\gamma, I_n), \quad \tilde{S}(\gamma) := \tilde{S}(\gamma, I_n), \quad S(\gamma) := S(\gamma, I_n), \quad (2.14)$$

$$\mu_i(\gamma) := \mu_i(\gamma, I_n), \quad \mathcal{LR}(\gamma) := \mathcal{LR}(\gamma, I_n), \quad \mathcal{T}(\gamma) := \mathcal{T}(\gamma, I_n). \quad (2.15)$$

In Section 6, we focus on the following special case of (2.2): we take $C = s_k[j]'$ so that

$\tilde{H}$ has the form

$$\tilde{H}_j(\delta, G) : s_k[j]'BG = \delta' \quad (2.16)$$

where δ is a $g \times 1$ fixed vector, $j \in \{1, \dots, k\}$, and $s_k[j] = (I_{1j}, \dots, I_{kj})'$ with $I_{ij} = 1$ if $i = j$ and $I_{ij} = 0$ if $i \neq j$. This yields simultaneous bands for the components of the $g \times 1$ vector

$$\beta_j(G) = [\beta_{1j}(G), \dots, \beta_{ij}(G), \dots, \beta_{gj}(G)]' := (s_k[j]'BG)', \quad j \in \{1, \dots, k\}. \quad (2.17)$$

This vector can be viewed as the coefficient of the j -th regressor, in the regression of YG on X . As above, we will refer to the $n \times 1$ vector $\beta_j(I_n)$ as β_j , *i.e.*

$$\beta_j = (\beta_{1j}, \dots, \beta_{ij}, \dots, \beta_{nj})' := \beta_j(I_n), \quad j \in \{1, \dots, k\}, \quad (2.18)$$

so β_j' is the j -th row of B , and β_{ij} is the coefficient of regressor k in the regression of Y_{it} on X_t , $t = 1, \dots, T$, which we refer to as the i -th regression equation, for $i = 1, \dots, n$.

When X includes a constant, β_1 is the vector of intercepts. Conformably, let

$$\hat{\beta}_j(G) = [\hat{\beta}_{1j}(G), \dots, \hat{\beta}_{ij}(G), \dots, \hat{\beta}_{gj}(G)]' := (s_k[j]'\hat{B}G)', \quad j \in \{1, \dots, k\}, \quad (2.19)$$

$$\hat{\beta}_j = (\hat{\beta}_{1j}, \dots, \hat{\beta}_{ij}, \dots, \hat{\beta}_{nj})' := \hat{\beta}_j(I_n) = (s_k[j]'\hat{B})', \quad j \in \{1, \dots, k\}. \quad (2.20)$$

To test linear restrictions of the form (2.16), one can use a (generalized) Hotelling-type statistic:

$$\Lambda_j(\delta, G) = \frac{[\hat{\beta}_j(G) - \delta]'(G'\hat{S}G)^{-1}[\hat{\beta}_j(G) - \delta]}{s_k[j]'(X'X)^{-1}s_k[j]}; \quad (2.21)$$

see Rao (1973, Chapter 8, section 8c) and Anderson (2003, Chapter 8). For the case of a scalar linear combination $v'\delta$, $\Lambda_j(\delta, G)$ is associated with a t -type statistic:

$$\Lambda_j(v'\delta, G) = t_j(v'\delta, G)^2, \quad t_j(v'\delta, G) := \frac{v'[\hat{\beta}_j(G) - \delta]}{\{(v'G'\hat{S}Gv)(s_k[j]'(X'X)^{-1}s_k[j])\}^{1/2}}. \quad (2.22)$$

This fact will be useful for inference on $v'\delta$.

Our proposed confidence sets will be constructed by inverting α -level tests of the form $\Lambda_j(\delta, G) \geq c_j(\alpha)$ over the possible values of δ . For $G = I_n$ (no equation transformation), we shall use the simplified notation:

$$\Lambda_j(\delta) := \Lambda_j(\delta, I_n) = \frac{(\hat{\beta}_j - \delta)'\hat{S}^{-1}(\hat{\beta}_j - \delta)}{s_k[j]'(X'X)^{-1}s_k[j]}. \quad (2.23)$$

The following theorem provides the link between $\Lambda_j(\delta, G)$ and the LR statistic $\mathcal{LR}(\gamma, G)$.

Theorem 2.1. *Under the null hypothesis (2.2) on model (2.1), the LR criterion $\mathcal{LR}(\gamma, G)$*

in (2.12) takes the form $\mathcal{LR}(\gamma, G) = T \ln(|I_c + \tilde{\Lambda}(\gamma, G)|)$ where

$$\tilde{\Lambda}(\gamma, G) = [C(X'X)^{-1}C']^{-1}(C\hat{B}G - D)(G'\hat{S}G)^{-1}(C\hat{B}G - D)'. \quad (2.24)$$

Theorem 2.1 shows that $\Lambda_j(\delta, G) = \tilde{\Lambda}(\gamma, G)$ when $C = s_k[j]'$ and $D = \delta'$. Applications to simultaneous inference on the components of $\beta_j(G)$ are discussed in Section 6.

3 Distributions of test criteria: baseline regression

We will now characterize the null distributions of the test statistics introduced in Section 2.2, under the baseline setting (2.1). Under the hypothesis (2.2) on (2.1), this can be done by characterizing the distribution of the eigenvalues $\mu_1(\gamma, G) \geq \dots \geq \mu_n(\gamma, G)$.

Theorem 3.1. *Under the null hypothesis $\tilde{H}$ in (2.2) on model (2.1), the vector of the non-zero roots of (2.10) is distributed (conditional on J) like the vector of the non-zero roots of the equivalent determinantal equations:*

$$\det\{(G'J)W'[\mathcal{M}_0(X, C) - \mu\mathcal{M}(X)]W(J'G)\} = 0, \quad (3.1)$$

$$\det\{(W\mathcal{G})'[\mathcal{M}_0(X, C) - \mu\mathcal{M}(X)](W\mathcal{G})\} = 0, \quad (3.2)$$

where $\mathcal{M}_0(X, C) := X(X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}C(X'X)^{-1}X'$, and $\mathcal{G}$ is the orthogonal $n \times g$ matrix of the eigenvectors associated with the non-zero eigenvalues of $J'GG'J$.

It is important to note here that the eigenvalue distribution depends on the unknown J matrix (which determines the covariance matrix of U_t) through the matrix $J'GG'J$. Since J can be high-dimensional, this allows for the presence of many nuisance parameters. There is no guarantee that the roots of (2.10) are pivotal even with $g = 1$, *i.e.* even if one focuses on a single equation or a single known linear combination of the endogenous variables (*e.g.*, by forming a portfolio). C and G also affect the distributions, but these are known matrices. We now give three general conditions under which the unknown J matrix disappears from the distribution of the eigenvalues.

Theorem 3.2. *Under the assumptions of Theorem 3.1, suppose that at least one of the following assumptions is satisfied: (i) $G = I_n$; (ii) G and J satisfy the separability condition (2.7); (iii) W satisfies the mixture assumption (2.4). Then (3.2) is equivalent to*

$$\det\{W'[\mathcal{M}_0(X, C) - \mu\mathcal{M}(X)]W\} = 0 \quad (3.3)$$

where: $\mathcal{W} = W$ if (i) holds; $\mathcal{W} = WG$ if (ii) holds; $\mathcal{W} = V\mathcal{Z}(g)$ if (iii) holds.

Condition (i) means that the restrictions tested apply *uniformly* to all the equations; condition (ii) represents linear restrictions on subset of equations along with a type of separability (or causal ordering) between two subvectors of disturbances; condition (iii) is a multivariate normal mixture assumption. For these special cases, we have: (i) the roots of (3.3) are invariant to B and J , and allow one to perform exact Monte Carlo tests p -values when W can be simulated [Dufour and Khalaf (2002)]; (ii) the associated null distributions depend on C but not on D , and depend on G only through its rank. Due to its importance, we spell out the null distribution of the Hotelling-type statistic $\Lambda_j(\delta, G)$ in the following corollary.

Corollary 3.1. *Under the null hypothesis $\tilde{H}_j$ [in (2.16)] on model (2.1), suppose that any one of the conditions (i), (ii) or (iii) of Theorem 3.2 is satisfied, and let $\mathcal{W}$ be defined as in (3.3). Then the statistic $\Lambda_j(\delta, G)$ [defined by (2.21)] is distributed like the random variable*

$$\bar{\Lambda}_j(\mathcal{W}) = \frac{s_k[j]'(X'X)^{-1}X'\mathcal{W}[\mathcal{W}'\mathcal{M}(X)\mathcal{W}]^{-1}\mathcal{W}'X'(X'X)^{-1}s_k[j]}{s_k[j]'(X'X)^{-1}s_k[j]}. \quad (3.4)$$

This corollary is a direct consequence of Theorem 3.2. The distribution of $\bar{\Lambda}_j(\mathcal{W})$ may be viewed as a *generalized Hotelling distribution*. A key property here is that the null distribution of $\bar{\Lambda}_j(\mathcal{W})$ does not depend on the tested value δ [as with usual t statistics in univariate linear regressions], so the corresponding critical points can be used to test any value of δ . We shall exploit this feature in Section 6 to invert the associated tests and build confidence sets.

4 Incomplete multivariate regression

In this section, we relax the assumption that (2.1) is complete, *i.e.* we allow for unobserved explanatory variables. We first consider a general MLR model, and then discuss applications to linear asset pricing models.

4.1 Missing regressors in a multivariate linear regression

Let us now reconsider (2.1) and incorporate an unobservable regressor Z as follows:

$$Y = XB + ZB_Z + U \Leftrightarrow Y_t = B'X_t + B_Z'Z_t + U_t, \quad t = 1, \dots, T, \quad (4.5)$$

where Z'_t denotes the t -th row of Z . As in Section 2, we focus on the conditional distribution of Y given the explanatory (or input) variables X and Z , and we assume that U is independent of X and Z with $E(U) = 0$. This covers the basic case where X and Z are non-random (fixed constants), as well as cases where X and Z are random but we focus on the conditional distribution of Y given $[X, Z]$. In linear regression models like (2.1), it is common practice to reparametrize the model in a way that makes parameters easier to interpret or estimate: this includes centering, standardization, and various forms of orthogonalization like principal component (or latent root) transformations [see Seber (1984, Chap. 5), Jackson (2003, Chap. 12-13), Jolliffe (2004, Chap. 8)]. In such cases, the parameters of the model naturally depend on the regressors, and thus may be interpreted as *ex post* (or *realized*) parameters. An important feature of *ex post* parameters is that they represent actual behavior of the data, rather than imaginary superpopulations.

To deal with the issue of missing explanatory variables, we propose to consider a reparametrized formulation of (4.5). By definition,

$$\mathcal{P}[X]Z = X(X'X)^{-1}X'Z = XB_{XZ}, \quad B_{XZ} := (X'X)^{-1}X'Z. \quad (4.6)$$

Without loss of generality, we can write:

$$\begin{aligned} Y &= XB + \mathcal{P}[X]ZB_Z + ZB_Z - \mathcal{P}[X]ZB_Z + U \\ &= X[B + B_{XZ}B_Z] + \mathcal{M}(X)ZB_Z + U \\ &= X\Theta + [\mathcal{M}(X)Z]B_Z + U, \quad \Theta := B + B_{XZ}B_Z. \end{aligned} \quad (4.7)$$

In the above reparametrization, the observed and non-observed components of Y are transformations of matrices whose columns are mutually orthogonal $[X$ and $\mathcal{M}(X)Z]$. The identity $\Theta = B + B_{XZ}B_Z$ is the *definition* of Θ (not an assumption). Further, it is easy to see that $\hat{B} = (X'X)^{-1}X'Y$ is an unbiased estimator of Θ , in the sense that $E[\hat{B} - \Theta | X, Z] = 0$, so Θ is identified (conditional of X and Z).

Θ can be interpreted as the effect of changing both X and Z , in a way that mimics the way X and Z are related in the regressor matrix $[X, Z]$. This is done by considering the projection $\mathcal{P}[X]Z$: if X_1 is an alternative value of the matrix X , the change $\Delta X = X_1 - X$ can be mapped (through a linear approximation of Z by X) to a change in Z given by $\Delta Z = Z_1 - Z = (X_1 - X)B_{XZ}$. The corresponding total change in Y (keeping U unchanged) is then

$$Y_1 - Y = (X_1 - X)B + (X_1 - X)B_{XZ}B_Z = (X_1 - X)\Theta. \quad (4.8)$$

Alternatively, if

$$Y_0 = XB + ZB_Z + U_0 \text{ and } Y_1 = X_1B + Z_1B_Z + U_1 \quad (4.9)$$

where $Z_1 - Z = (X_1 - X)B_{XZ}$, we can write [under model (4.5)]:

$$E[Y_1 | X_1, Z_1] - E[Y_0 | X, Z] = (X_1 - X)\Theta, \quad (4.10)$$

so $(X_1 - X)\Theta$ represents a total effect where the missing regressors Z are allowed to change in a way similar to the one suggested by the linear structure of $[X, Z]$.

Θ may also be viewed as a pseudo-true value, with three important characteristics. *First*, Θ is sample dependent, which may be viewed a “realized” or “ex-post” parametrization. *Second*, Θ integrates comovements between the observed and left-out regressors (measured by B_{XZ}). *Third*, the extent of these comovements is affected by B_Z , which captures the contribution of the missing regressors to the regression. The interpretation of any pseudo-true value is sample dependent. Θ emerges as a natural inference target in many contexts.

4.2 Incomplete linear asset pricing model

The asset pricing application we consider in Section 7 provides a useful framework to illustrate the above approach. The model in question takes the form (2.1), where Y_{it} is an (excess) return on a financial security, and X includes ι_T , and (excess) returns on a given set of observed risk factors. In this context: (i) the above defined $n \times 1$ vector β_1 , which collects the coefficients of the intercept over all considered securities, is taken for *alpha*, *i.e.* the risk-adjusted excess return, and (ii) each one of the vectors β_j for $j = 2, \dots, k$ is regarded as the *beta*, *i.e.* the risk component relative to each of the factors.

The fundamental notion of *alpha* implies outperformance, adjusted for risk in its many forms. However, the definition of the intercept within any regression makes it clear which *alpha* is tied to a given set of risk factors. This means that a regression-based *alpha* may (partly) pick up exposure to an unmeasured source of risk which is correlated with the considered factors. The same applies to regression-based measures of *betas*. (4.5) formalizes this argument: unless $X'Z = 0$ or $B_Z = 0$, the regression identifies Θ as defined in (4.7), rather than B . Mimicking portfolios can also lead to similar effects; see Kleibergen and Zhan (2018) and Beaulieu et al. (2023).

Regression *betas* may nevertheless remain valid measures of exposure, in the sense that they reflect risk irrespective of the unobserved component they might embed, even if the

underlying source of risk may not be strictly singled out. The interpretation of β_1 raises deeper concerns, since the assumption of complete risk-adjustment which underlies *alpha* in the strict sense, precludes missing factors. Empirical patterns which look anomalous can thus be explained through exposure to some understandable common risk factor ignored by previous models. As emphasized in Section 1, such lines of reasoning have enduring support which validates reliance on (4.7) to analyze the risk-return profile for the particular risk scenario captured by the realized sample of factors.

In asset pricing, realized parameters are commonly analyzed, particularly from a general equilibrium perspective. For example, MacKinlay (1995) and MacKinlay and Pastor (2000) capture deviations from the CAPM through an unobserved factor which annuls mispricing *alphas*; see also Beaulieu et al. (2023) and Beaulieu et al. (2025). The “realized” attribute of Θ thus provides a further credible direction to interpret identifiable parameters. We thus proceed to analyze the properties of standard tests applied within (4.5).

We now assume that conditional on X and Z , the normality assumption holds, so $U_t \stackrel{\text{iid}}{\sim} N[0, JJ']$. In view of the above discussion, we will now take Θ as the parameter of interest, and consider restrictions of the form:

$$H_\Theta : C\Theta = D. \quad (4.11)$$

In the context of restriction (2.2), we can assume $G = I_n$ without loss of generality because of the Gaussian error assumption.

As with complete regressions, we study the roots $\mu_1(\gamma) \geq \dots \geq \mu_n(\gamma)$ of the determinantal equation (2.10), here with $G = I_n$ conforming with H_Θ . The multivariate test statistics considered above involve elementary symmetric functions of these roots [see Olkin and Tomskey (1981)]. Specifically, for $p \leq q$,

$$\lambda_{p,q}[\mu_1(\gamma), \dots, \mu_n(\gamma)] := \sum_{\mu_{i_1} < \dots < \mu_{i_p}} \mu_{i_1}^*(\gamma) \cdots \mu_{i_q}^*(\gamma), \quad \mu_i^*(\gamma) = \frac{T-k}{c} \mu_i(\gamma), \quad (4.12)$$

denotes the p -th elementary symmetric function of the largest q roots. For example,

$$\lambda_{1,n}(\cdot) = \sum_{i=1}^n \mu_i^*(\gamma), \quad \lambda_{n,n}(\cdot) = \prod_{i=1}^n \mu_i^*(\gamma), \quad \lambda_{1,1}(\cdot) = \sup\{\mu_1^*(\gamma), \dots, \mu_n^*(\gamma)\}. \quad (4.13)$$

In what follows, we show that inference based on $\lambda_{p,q}(\mu_1(\gamma), \dots, \mu_n(\gamma))$ remains valid (in the sense that the true level is not larger than its nominal level) despite the fact that

model (4.5) is incomplete. Two sets of algebraic properties are used to establish these results. The first one includes the following decompositions under (4.11):

$$C\hat{B} - D = C(X'X)^{-1}X'U, \quad \hat{S} = (ZB_Z + U)' \mathcal{M}(X)(ZB_Z + U), \quad (4.14)$$

where we see that missing regressors appear in $\hat{S}$. The second set of results is interesting in its own right, so we explicit it in the following proposition, which builds on Olkin and Tomsy (1981, Theorem 4.6) and the Courant-Fischer theorem [see *e.g.* Horn and Johnson (1990, Theorem 4.2.11)].

Proposition 4.1. *In the context of model (2.1) along with the notation in (2.2)-(2.20), let $\mu_1(\gamma), \dots, \mu_n(\gamma)$ be the roots of (2.10) where γ is defined in (2.3). Then*

$$\mu_1(\gamma) = \frac{c}{T-k} \sup_v \mathcal{F}_v(\gamma), \quad (4.15)$$

$$\mu_i(\gamma) = \frac{c}{T-k} \sup_{G_i} \inf_{v_i} \mathcal{F}_{v_i}(\gamma, G_i), \text{ for } i = 2, \dots, n, \quad (4.16)$$

where the optima are taken over the set of non-zero $n \times 1$ vectors v , the set of $n \times i$ matrices G_i of rank i , and the set of non-zero $i \times 1$ vectors v_i , and

$$\mathcal{F}_v(\gamma) = \frac{v'[S(\gamma) - \hat{S}]v/c}{v'\hat{S}v/(T-k)}, \quad \mathcal{F}_{v_i}(\gamma, G_i) = \frac{v_i'(G_i'[S(\gamma) - \hat{S}]G_i)v_i/c}{v_i'(G_i'\hat{S}G_i)v_i/(T-k)}, \quad (4.17)$$

$$S(\gamma) = \hat{S} + (C\hat{B} - D)' [C(X'X)^{-1}C']^{-1} (C\hat{B} - D). \quad (4.18)$$

(4.15)-(4.18) express $\mu_1(\gamma), \dots, \mu_n(\gamma)$ as extrema of properly defined univariate F -type statistics.⁴ We now give the following dominance result on the distribution of test statistics based on symmetric functions of the roots of the determinantal equation (2.10).

Theorem 4.1. *In the context of model (4.5), suppose that the Gaussian disturbance assumption (2.4) with $V = I_T$ holds conditionally on X and Z , and let $\mu_1(\gamma) \geq \dots \geq \mu_n(\gamma)$ be the roots defined in Proposition 4.1. Then, under the null hypothesis H_Θ in (4.11), we have:*

$$\Pr[\lambda_{p,q}(\mu_1(\gamma), \dots, \mu_n(\gamma)) \geq x] \leq \Pr[\lambda_{p,q}(\bar{\mu}_1(C), \dots, \bar{\mu}_n(C)) \geq x] \quad (4.19)$$

⁴In the next section, we use similar results which relate Hotelling statistics to properly defined t-type statistics, which help interpret our proposed simultaneous confidence sets.

where $\bar{\mu}_1(C) \geq \dots \geq \bar{\mu}_n(C)$ are the non-zero roots of the determinantal equation (3.3) with $\mathcal{W} = W$, and $\lambda_{p,q}(\cdot)$ denotes any elementary symmetric function as defined in (4.12).

The bound in (4.19) is free of nuisance parameter parameters, and depends on the normality assumption on U_t . This is not necessarily restrictive, since the effective disturbance term in (4.5) is $(B'_Z Z_t + U_t)$ where the component $B'_Z Z_t$ is unobserved. This structure can conveniently capture deviations from normality.

5 Asymptotic validity

The above finite-sample results hold under restrictive assumptions. However, it is easy to see that the corresponding tests remain asymptotically valid under standard limit assumptions. In this section, we spell out standard generic assumptions under which this will be the case. Suppose that the following assumptions hold: as $T \rightarrow \infty$,

$$\frac{1}{T} X'U \xrightarrow{p} 0, \quad \frac{1}{T} X'X \xrightarrow{p} \Sigma_X, \quad \frac{1}{T} U'U \xrightarrow{p} \Sigma_U > 0, \quad (5.20)$$

$$\frac{1}{\sqrt{T}} \text{vec}(X'U) \xrightarrow{d} N[0, \Sigma_U \otimes \Sigma_X]. \quad (5.21)$$

Then, both Wald and LR statistics follow a $\chi^2(cg)$ limiting null distribution under model (2.1); see Berndt and Savin (1977), Gouriéroux et al. (1995) and Dufour and Khalaf (2002). Furthermore, the roots $T\mu_1(\gamma, G), \dots, T\mu_n(\gamma, G)$ follow the same null distribution as the roots of $T\bar{S}(C, G)$, where

$$\bar{S}(C, G) = G'U'X (X'X)^{-1} C'[C(X'X)^{-1}C']^{-1} C (X'X)^{-1} X'UG[G'U'\mathcal{M}(X)UG]^{-1}.$$

Following the derivations of Anderson (2003, Chapter 8), the limiting distribution of the roots of $T\bar{S}(C, G)$ is the distribution of the roots of a central Wishart with dimension g , degrees-of-freedom c , and scale I_g . Based on Dufour (2006), this generalizes the validity of Monte Carlo tests based on these roots, for distributional assumptions that preserve (5.20)-(5.21). This includes the mixtures assumption (2.4) with random V : conditional on V , both Wald and LR statistics follow a $\chi^2(cg)$ limiting null distribution. Since this distribution does not depend on V , the result also holds unconditionally.

In the incomplete regression (4.5) where we set $G = I_n$, the following additional assumptions are required:

$$\frac{1}{T} B'_Z Z'U \xrightarrow{p} 0, \quad \frac{1}{T} B'_Z Z'Z B_Z \xrightarrow{p} \Sigma_Z > 0. \quad (5.22)$$

Under these assumptions and relaxing normality, (A.22) and (A.24) extend asymptotically for the Wald counterpart of the $\mathcal{F}_v(\gamma)$ statistic [see Dufour and Nguyen (2015)] in the following sense: the limiting distribution of $[cT/(T-k)]\bar{\mathcal{F}}T - k_v(C)$ is $\chi^2(c)$, and

$$\lim_{T \rightarrow \infty} \Pr \left[\frac{cT}{T-k} \mathcal{F}_v(\gamma) \geq x \right] \leq \lim_{T \rightarrow \infty} \Pr \left[\frac{cT}{T-k} \bar{\mathcal{F}}_v(C) \geq x \right]. \quad (5.23)$$

It follows that

$$\lim_{T \rightarrow \infty} \Pr [T \mu_j(\gamma) \geq x] \leq \lim_{T \rightarrow \infty} \Pr [T \bar{\mu}_j(C) \geq x] \quad (5.24)$$

where the limiting distribution associated with the right-hand side is that of the roots of $T\bar{S}(C, I_n)$, a central Wishart with dimension n degrees-of-freedom c and scale I_n . This establishes the asymptotic counterpart of Theorem 4.1.

6 Simultaneous inference on factor loadings

In this section, we build simultaneous confidence bands for the components of vector $\beta_j = \beta_j(G)$, associated with the j -th regressor in different equations, where $1 \leq j \leq k$ [see (2.17)]. We proceed by inversion of Hotelling-type tests $\Lambda_j(\delta, G)$ of $\tilde{H}_j(I_n, \delta)$ [see (2.16) - (2.23)] over δ . Under the assumptions of Corollary 3.1 [with (i), (ii) or (iii)], the null distribution of $\Lambda_j(\delta, G)$ does not depend on δ nor any nuisance parameter, so that we can write:

$$\Lambda_j(\delta, G) \sim \bar{\Lambda}(X, G) \quad (6.1)$$

where the generalized Hotelling distribution $\bar{\Lambda}(X, G)$ is characterized by the representation (3.4). If $\bar{\Lambda}_\alpha := \bar{\Lambda}_\alpha(X, G)$ satisfies $\Pr[\bar{\Lambda}(X, G) \geq \bar{\Lambda}_\alpha] = \alpha$, for $0 < \alpha < 1$, the set

$$\begin{aligned} \text{CS}_\alpha(j) &:= \{\delta : \Lambda_j(\delta, G) \leq \bar{\Lambda}_\alpha\} \\ &= \{\delta : [\hat{\beta}_j(G) - \delta]'(G' \hat{S} G)^{-1} [\hat{\beta}_j(G) - \delta] \leq (s_k[j]'(X'X)^{-1} s_k[j]) \bar{\Lambda}_\alpha\} \end{aligned} \quad (6.2)$$

is a confidence set with level $1 - \alpha$ for $\beta_j(G) = (s_k[j]'BG)'$. Clearly, $\text{CS}_\alpha(j)$ represents an ellipsoid in $\mathbb{R}^g$. In the standard case (i) [Gaussian errors with $V = I_T$], the distribution $(\tau_n/n)\bar{\Lambda}(X, G)$ is identical with the central Fisher distribution $F(n, \tau_n)$ where $\tau_n = T - k - n + 1$; see Rao (1973, Section 8c.4). In this case, we have:

$$\bar{\Lambda}_\alpha(X, I_n) = (n/\tau_n)F_\alpha(n, \tau_n). \quad (6.3)$$

More generally, if the assumptions of Corollary 3.1 hold and $\mathcal{W}$ can be simulated, $\bar{\Lambda}_\alpha$ can be replaced by a simulated critical value based on the pivotal statistic (3.4). This simulated

critical value can be made arbitrarily accurate by using a large number of replications. Under the limit assumptions of Section 5, the confidence set $\text{CS}_\alpha(\beta_j)$ is also asymptotically valid. For T large, the critical value $(n/\tau_n)F_\alpha(n, \tau_n)$ can be approximated by the chi-square critical value $T\chi_\alpha^2(n)$.

$\text{CS}_\alpha[\beta_j(G)]$ is a joint confidence set for the coefficients of a given explanatory variable (the j -th regressor) across different (non-independent) equations. Simultaneous inference on various (possibly nonlinear) transformations of $\beta_j(G)$ can be obtained through a projection approach [Dufour (1989, 1997, 2005, 2007)]. For $\mathcal{F}$ a family of functions $f : \mathbb{R}^g \rightarrow \mathbb{R}$, the implication

$$\delta \in \text{CS}_\alpha(j) \Rightarrow \{h(\delta) \in f(\text{CS}_\alpha(j)) \text{ for all } f \in \mathcal{F}\} \quad (6.4)$$

where $g(\text{CS}_\alpha(j))$ represents the image set of $\text{CS}_\alpha(j)$, allows us to write

$$\Pr [g(\beta_j(G)) \in f(\text{CS}_\alpha(j)) \text{ for all } f \in \mathcal{F}] \geq \Pr [\beta_j(G) \in \text{CS}_\alpha] \geq 1 - \alpha. \quad (6.5)$$

In other words, the confidence sets $f(\text{CS}_\alpha(j))$ are simultaneous at level $1 - \alpha$ for all parameter transformations $f(\beta_j(G))$ in $\mathcal{F}$. Of course, this can be applied to linear transformations $f(\beta_j) = v'\beta_j$.

Due to its importance, we now give a more explicit simultaneity result for linear combinations $v'\beta_j(G)$ – which includes individual coefficients of $\beta_j(G)$. It is based on observing that Hotelling-type statistics can be viewed as maxima of t -type statistics.⁵

Theorem 6.1. *Under model (2.1) with the notations (2.2)-(2.20), the Hotelling-type statistics $\Lambda_j(\delta, G)$ and $\Lambda_j(\delta)$ in (2.21)-(2.23) satisfy: for $1 \leq j \leq k$,*

$$\Lambda_j(\delta, G) = \sup_v \{t_j^2(v; \delta, G)\} \quad (6.6)$$

where $t_j(v'\delta, G) := v'[\hat{\beta}_j(G) - \delta]/\hat{\sigma}_j(v)$, $\hat{\sigma}_j(v) := \{(v'G'\hat{S}Gv)(s_k[j]'(X'X)^{-1}s_k[j])\}^{1/2}$, and the optima are taken over any g -dimensional non-zero vector v .

The result of Theorem 6.1 provides valid distributional bounds for any number of linear restrictions of the form $v'\beta_j(G) - \delta = 0$ over different values of ν . This yields intervals of the form:

$$\text{CS}_\alpha(j, v, G) := \{v'\hat{\beta}_j(G) - v'\delta \mid \leq \bar{\Lambda}_\alpha^{1/2}\hat{\sigma}_j(v)\} \quad (6.7)$$

⁵See Scheffé (1959), Savin (1984) and Dufour (1989) in the context of univariate regression.

which satisfy the simultaneous property

$$\Pr[v' \beta_j(G) \in \text{CS}_\alpha(j, v, G) \text{ for all } v \in \mathbb{R}^n] \geq 1 - \alpha. \quad (6.8)$$

We can also build simultaneous confidence intervals for the individual coefficients of each vector $\beta_j(G) := (\beta_{1j}, \dots, \beta_{gj})'$ [see (2.18)] by considering different selection vectors $v \in \{s_g[i], i = 1, \dots, g\}$ where $s_g[i] = (I_{1i}, \dots, I_{gi})'$ [see (2.16)].

Finally, due to the structure of the Hotelling-type statistics, the dominance properties established in Section 4 for incomplete multivariate regressions also hold for the above simultaneous confidence sets.

7 Empirical analysis

We use the above methodology to analyze risk-return regressions for **CBMFs**. In our regressions, Y_{it} are returns on a set of seven excess CMBFs described in Appendix B. X includes ν_T and (excess) returns on well-known risk benchmarks. Disturbances are assumed to be Gaussian as in (2.5) or Student- $t(\kappa)$ as in (2.4) with (2.6), with unknown degrees-of-freedom κ . As benchmarks, we consider: (i) the single-factor market return based on the Capital Asset Pricing Model [**CAPM**, Sharpe (1964), Lintner (1965)]; (ii) the Quadratic CAPM [**QCAPM**, Kraus and Litzenberger (1976), Barone-Adesi, Gagliardini and Urga (2004)], which incorporates the excess market return and a mimicking portfolio for its squared return as a proxy for coskewness; and (iii) the three-factor Fama-French benchmark [Fama and French (1993)] with the excess market return, Small-minus-Big (SMB) and High-minus-Low (HML) portfolios.

Although severe, risk triggers arising from natural disasters are thought to create much less stress on equity markets. Furthermore, the occurrence and extent of events which determine catastrophe-linked default risk are not affected by common determinants of bond markets [Herrmann and Hibbeln (2023)]. While compensating investors for risk bearing, returns on Cat bonds are thus expected to comove weakly with other capital markets. This suggests that CMBFs can, potentially, yield positive expected returns and provide a hedge against market shocks. The focus of our analysis is to quantify these claims statistically.⁶ For this purpose, estimates of both return sources (*alpha* and *betas*)

⁶For technical reports on this perspective and information on the underlying industry and market, see Polacek (2018), Braun and Kousky (2021), and American Academy of Actuaries (2022). References on related empirical analyses are reviewed below.

Table 1: CAPM

CBMFs	11/2004 - 03/2011	
	CS(a)	CS(b _M)
Panel A: Gaussian		
1	(-0.00611 , 0.00359)	(-0.06210 , 0.13548)
2	(-0.00463 , 0.00461)	(-0.06443 , 0.12390)
3	(-0.00476 , 0.00552)	(-0.06323 , 0.14646)
4	(-0.00587 , 0.00384)	(-0.06557 , 0.13241)
5	(-0.00332 , 0.00340)	(-0.01117 , 0.12585)
6	(-0.00366 , 0.00395)	(-0.01420 , 0.14094)
7	(-0.00334 , 0.00485)	(-0.02370 , 0.14328)
Panel B: Student-$t(\eta)$		
1	(-0.00610 , 0.00358)	(-0.07034 , 0.14371)
2	(-0.00462 , 0.00460)	(-0.07228 , 0.13175)
3	(-0.00475 , 0.00551)	(-0.07197 , 0.15520)
4	(-0.00586 , 0.00383)	(-0.07382 , 0.14066)
5	(-0.00331 , 0.00340)	(-0.01688 , 0.13156)
6	(-0.00365 , 0.00394)	(-0.02066 , 0.14741)
7	(-0.00333 , 0.00484)	(-0.03066 , 0.15024)
CBMFs	04/2011 - 08/2014	
	Panel C: Gaussian	
1	(0.00011 , 0.00480)*	(-0.05991 , 0.06988)
2	(0.00092 , 0.00528)*	(-0.06414 , 0.05660)
3	(0.00103 , 0.00545)*	(-0.06315 , 0.05925)
4	(0.00056 , 0.00523)*	(-0.06057 , 0.06874)
5	(-0.00020 , 0.00295)	(-0.04805 , 0.03908)
6	(0.00024 , 0.00342)*	(-0.05210 , 0.03605)
7	(0.00050 , 0.00356)*	(-0.04719 , 0.03775)
Panel D: Student-$t(\eta)$		
1	(0.00005 , 0.00486)*	(-0.06058 , 0.07056)
2	(0.00086 , 0.00534)*	(-0.06476 , 0.05723)
3	(0.00097 , 0.00551)*	(-0.06379 , 0.05988)
4	(0.00050 , 0.00530)*	(-0.06124 , 0.06940)
5	(-0.00024 , 0.00299)	(-0.04850 , 0.03953)
6	(0.00019 , 0.00346)*	(-0.05256 , 0.03650)
7	(0.00045 , 0.00360)*	(-0.04763 , 0.03819)

Note – Reported intervals are exact confidence sets (CS) estimated for the degrees of freedom parameter, η , from the *i.i.d.* jointly multivariate Student- $t(\eta)$ error distribution. Estimated CS contain only values of η for which the joint null hypothesis of no excess skewness and kurtosis (relative to the Student- t) is not rejected. Specifically, a given value of η is retained in the 5%-level CS only if the associated p -value exceeds 5%.

Table 2: QCAPM

CBMFs	11/2004 - 03/2011		
	CS(a)	CS(b _M)	CS(b _S)
Panel A: Gaussian			
1	(-0.00625 , 0.00366)	(-0.08872 , 0.15594)	(-0.17914 , 0.19535)
2	(-0.00475 , 0.00468)	(-0.08889 , 0.14435)	(-0.17323 , 0.18377)
3	(-0.00475 , 0.00574)	(-0.07753 , 0.18162)	(-0.22581 , 0.17088)
4	(-0.00601 , 0.00391)	(-0.09185 , 0.15331)	(-0.18054 , 0.19472)
5	(-0.00339 , 0.00348)	(-0.02768 , 0.14202)	(-0.12944 , 0.13032)
6	(-0.00375 , 0.00402)	(-0.03336 , 0.15879)	(-0.14532 , 0.14879)
7	(-0.00338 , 0.00498)	(-0.03967 , 0.16705)	(-0.16848 , 0.14794)
Panel B: Student-$t(\eta)$			
1	(-0.00619 , 0.00360)	(-0.09884 , 0.16606)	(-0.19533 , 0.21153)
2	(-0.00470 , 0.00463)	(-0.09853 , 0.15400)	(-0.18866 , 0.19920)
3	(-0.00469 , 0.00569)	(-0.08825 , 0.19235)	(-0.24295 , 0.18802)
4	(-0.00596 , 0.00386)	(-0.10200 , 0.16346)	(-0.19676 , 0.21094)
5	(-0.00336 , 0.00344)	(-0.03470 , 0.14905)	(-0.14066 , 0.14154)
6	(-0.00371 , 0.00398)	(-0.04131 , 0.16673)	(-0.15803 , 0.16149)
7	(-0.00334 , 0.00494)	(-0.04822 , 0.17560)	(-0.18215 , 0.16161)
CBMFs	04/2011 - 08/2014		
	Panel C: Gaussian		
1	(0.00001 , 0.00515)*	(-0.06168 , 0.07768)	(-0.13301 , 0.10321)
2	(0.00092 , 0.00568)*	(-0.06341 , 0.06556)	(-0.13326 , 0.08536)
3	(0.00107 , 0.00588)*	(-0.06173 , 0.06877)	(-0.13770 , 0.08351)
4	(0.00051 , 0.00562)*	(-0.06132 , 0.07727)	(-0.13672 , 0.09820)
5	(-0.00031 , 0.00314)	(-0.05046 , 0.04328)	(-0.08386 , 0.07502)
6	(0.00016 , 0.00365)*	(-0.05347 , 0.04121)	(-0.08963 , 0.07086)
7	(0.00041 , 0.00378)*	(-0.04874 , 0.04254)	(-0.08537 , 0.06934)
Panel D: Student-$t(\eta)$			
1	(-0.000001 , 0.00516)	(-0.06551 , 0.08151)	(-0.13292 , 0.10312)
2	(0.000910 , 0.00569)*	(-0.06696 , 0.06911)	(-0.13318 , 0.08528)
3	(0.001050 , 0.00589)*	(-0.06532 , 0.07236)	(-0.13762 , 0.08342)
4	(0.000490 , 0.00563)*	(-0.06514 , 0.08108)	(-0.13663 , 0.09812)
5	(-0.000320 , 0.00315)	(-0.05304 , 0.04586)	(-0.08381 , 0.07497)
6	(0.000150 , 0.00366)*	(-0.05608 , 0.04381)	(-0.08957 , 0.07080)
7	(0.000410 , 0.00379)*	(-0.05125 , 0.04505)	(-0.08532 , 0.06929)

Note – Estimated model: quadratic CAPM; see notes to table 1.

Table 3: Fama-French

CBMFs	11/2004 - 03/2011			
	CS(a)	CS(b _M)	CS(b _{SMB})	CS(b _{HML})
Panel A: Gaussian				
1	(-0.0062 , 0.0037)	(-0.0795 , 0.1629)	(-0.2306 , 0.2050)	(-0.1499 , 0.1392)
2	(-0.0047 , 0.0047)	(-0.0834 , 0.1478)	(-0.2169 , 0.1986)	(-0.1387 , 0.1370)
3	(-0.0048 , 0.0057)	(-0.0913 , 0.1658)	(-0.2386 , 0.2235)	(-0.1369 , 0.1697)
4	(-0.0060 , 0.0039)	(-0.0825 , 0.1604)	(-0.2344 , 0.2021)	(-0.1493 , 0.1403)
5	(-0.0034 , 0.0035)	(-0.0234 , 0.1445)	(-0.1720 , 0.1297)	(-0.0952 , 0.1049)
6	(-0.0037 , 0.0041)	(-0.0288 , 0.1610)	(-0.1987 , 0.1424)	(-0.1027 , 0.1236)
7	(-0.0034 , 0.0049)	(-0.0408 , 0.1641)	(-0.1986 , 0.1696)	(-0.1180 , 0.1263)
Panel B: Student-$t(\eta)$				
1	(-0.0063 , 0.0037)	(-0.0865 , 0.1699)	(-0.2331 , 0.2075)	(-0.162 , 0.1514)
2	(-0.0048 , 0.0047)	(-0.09 , 0.1544)	(-0.2192 , 0.2009)	(-0.1503 , 0.1486)
3	(-0.0049 , 0.0057)	(-0.0987 , 0.1732)	(-0.2411 , 0.2261)	(-0.1498 , 0.1826)
4	(-0.006 , 0.004)	(-0.0895 , 0.1673)	(-0.2368 , 0.2045)	(-0.1615 , 0.1525)
5	(-0.0034 , 0.0035)	(-0.0282 , 0.1492)	(-0.1736 , 0.1314)	(-0.1036 , 0.1133)
6	(-0.0037 , 0.0041)	(-0.0342 , 0.1665)	(-0.2006 , 0.1443)	(-0.1122 , 0.1331)
7	(-0.0034 , 0.005)	(-0.0466 , 0.1699)	(-0.2006 , 0.1717)	(-0.1282 , 0.1366)
CBMFs	04/2011 - 08/2014			
	Panel C: Gaussian			
1	(0.0004 , 0.0052)*	(-0.0816 , 0.0599)	(-0.0799 , 0.1646)	(-0.0752 , 0.1595)
2	(0.0011 , 0.0056)*	(-0.0829 , 0.0504)	(-0.0820 , 0.1483)	(-0.0773 , 0.1438)
3	(0.0012 , 0.0058)*	(-0.0818 , 0.0535)	(-0.0854 , 0.1485)	(-0.0785 , 0.1460)
4	(0.0009 , 0.0056)*	(-0.0826 , 0.0577)	(-0.0763 , 0.1664)	(-0.0736 , 0.1593)
5	(-0.0003 , 0.0031)	(-0.0553 , 0.0436)	(-0.0849 , 0.0861)	(-0.0748 , 0.0894)
6	(0.0002 , 0.0035)*	(-0.0588 , 0.0413)	(-0.0882 , 0.0849)	(-0.0769 , 0.0892)
7	(0.0004 , 0.0037)*	(-0.0545 , 0.0420)	(-0.0816 , 0.0852)	(-0.0734 , 0.0867)
Panel D: Student-$t(\eta)$				
1	(0.0004 , 0.0052)*	(-0.0826 , 0.0609)	(-0.0814 , 0.1661)	(-0.0779 , 0.1622)
2	(0.0011 , 0.0056)*	(-0.0838 , 0.0514)	(-0.0834 , 0.1497)	(-0.0798 , 0.1463)
3	(0.0012 , 0.0058)*	(-0.0828 , 0.0545)	(-0.0868 , 0.1499)	(-0.0811 , 0.1485)
4	(0.0009 , 0.0056)*	(-0.0837 , 0.0588)	(-0.0777 , 0.1678)	(-0.0763 , 0.162)
5	(-0.0003 , 0.0031)	(-0.0561 , 0.0443)	(-0.0859 , 0.0871)	(-0.0766 , 0.0912)
6	(0.0001 , 0.0035)*	(-0.0596 , 0.042)	(-0.0892 , 0.0859)	(-0.0788 , 0.0911)
7	(0.0004 , 0.0037)*	(-0.0552 , 0.0427)	(-0.0826 , 0.0862)	(-0.0752 , 0.0885)

Note – Estimated model: Fama-French; see notes to table 1.

are required along with, crucially, estimates of associated uncertainty. Our methodology is distinct in that we collect the set of all parameters not rejected by joint significance tests across funds. This provides comprehensive information about the actual magnitude and direction of benefits, for the considered funds as a whole.

Our analysis focuses on the two sub-periods: 11/2004-03/2011 and 04/2011-12/2015 to exploit the effect of the Tohoku earthquake and tsunami which struck Japan in 2011, which we aim to contrast with those arising from seasonal downturn scenarios (*e.g.* hurricane season effects) and the stock market crash of 2008. Results are reported in tables 1-3, where confidence sets for each regressor have a 5% level simultaneously across the considered funds. Fund numbers in these tables refer to CBMFs described in the Appendix. Gaussian-based confidence bands are provided in Panel A and those based on Student- $t(\kappa)$ in Panel B. In the latter case, reported bands are the widest interval, obtained by simulation, over $2 \leq \kappa \leq 50$. Findings can be summarized as follows.

First, confidence bands do not vary across the considered error distributions: the bands vary little with κ and we find no noticeable difference with the Gaussian bands. Results (unreported) including the multivariate Cauchy distribution with $\kappa = 1$ lead to qualitatively similar results. This is useful for interpreting rejections when the regression is treated as incomplete.

Second, confidence bands for outperformance do not vary across models. This result is worth pointing out, given the arguments of Ardia et al. (2024) on comparing model *alphas* to their baseline CAPM counterparts. As for market *betas*, bands are slightly wider when skewness preference or size and book-to-market factors are accounted for. The addition of factors in our small samples can explain such differences, which at any rate remain rather negligible in economic terms.

Third, confidence bands for all factor sensitivities across all considered models cover zero and suggest a rather limited exposure to fluctuations in equity markets. For concreteness, across funds, the most negative lower (positive upper) bands do not reach -0.10 (0.20) for market *betas*, -0.23 (0.19) for coskewness *betas*, -0.25 (0.23) for size *betas* and -0.016 (0.18) book-to-market value *betas*.

Fourth, our key result is that following the 2011 earthquake, confidence bands for outperformance across all considered models are positive for all but one fund. We can interpret the economic importance of this finding: before the earthquake, confidence bands for *al-*

pha cover the -0.0063 to 0.0057 range; in contrast, set estimates for *alpha* post 2011 are positive for most funds, reaching up to 0.0059. While most lower bands on *betas* barely change post 2011, most upper bands are closer to zero after 2011. One way to view this finding, as suggested by Carhart, Cheah, De Santis, Farrell and Litterman (2014), is that extreme downside shocks affect tail-protection expectations, since extreme loss potential remains unattractive notwithstanding diversification appeal.

To put this finding into broader context, the literature suggests several channels through which adverse natural disasters can influence the Cat bond market. The first stems from the risk management practices of firms that issue a Cat bond. In particular, Hagendorff, Hagendorff, Keasey and Gonzalez (2014) find that: (i) actual and expected catastrophe losses increase the likelihood of firms to securitize, and (ii) catastrophe securitization leads firms to increase their exposure to the same type of risk they had previously hedged. Another channel relates to liquidity. For instance, Herrmann and Hibbeln (2023) find that increases in default likelihood stemming from expected seasonal stress (*e.g.* the US hurricane season) yield important reduction in secondary market trading, and (ii) liquidity is strongly priced, and more so for risky Cat bonds. A third channel involves the reputation of Cat bond issuers, which can affect risk premiums. In particular, Götze and Gürtler (2020) find that serious events alter the effects of the sponsor’s financial strength on premiums. Lastly, capital market linkages also constitute a credible - albeit debatable - channel. Before the default of Lehman Brothers in 2008, Cat bond structures included total return swaps that exposed these bonds to counterparty risk and thus, directly, to financial markets downturns [Herrmann and Hibbeln (2023)]; for further analyses that qualify market neutrality, see Cummins and Weiss (2009), Braun (2016), Braun, Ben Amar and Eling (2019), Beaulieu et al. (2023), and references therein. Whether such effects would translate into a positive *alpha* consistently is however unclear.

Considering that we found no evidence of outperformance prior to 2011, our view is that our finding of a positive *alpha* thereafter stems from the historically extreme nature of the Tohoku earthquake. This earthquake is the largest recorded in the history of Japan and the third-largest recorded worldwide since 1900; it also caused a devastating tsunami that led to an enormous economic and humanitarian crisis. These disasters set off the default of the Muteki Catastrophe bond with investors losing the entire principal of USD \$300

million as a consequence.⁷ It is important to note that our pre-2011 sample covers several hurricanes in the US including Katrina in 2005 which is one of the costliest on record. Despite the damages these severe events have caused, we find that their effects on CBMF *alphas* are muted. This suggests that unfamiliar tail shocks need to be disentangled from familiar disasters even if these are unlikely. Said differently, the consequences of extreme disasters can materially differ from those of regular or foreseeable stress scenarios. In this sense, seasonal events such as hurricanes are predictable.

A few related studies confirm this argument. For example, Zhao and Yu (2020), find that the primary market for Cat bonds integrates expectations about “well-defined” natural catastrophes; see also Herrmann and Hibbeln (2023) on seasonality in the secondary market. Bourdeau-Brien and Kryzanowski (2020) suggest that catastrophic events affect risk aversion depending on the “perceived level” of disastrous consequences; specifically, they find that catastrophic losses comparable to those experienced in the past have no impact on risk aversion, in contrast to losses that stand-out with reference to past experience. On the Muteki catastrophe itself, Akesaka (2019) reports that it has affected investors’ time preferences, through the present bias rather than the discount factor. Interestingly, we find that the 2008 financial crisis does not produce similar changes. As emphasized above, evidence on capital market linkages for Cat bonds remains mixed, which may explain our finding. As shown by Beaulieu et al. (2023), the 2008 financial crisis can weaken diversification potentials when the latter are gauged from an equilibrium perspective, *i.e.* when one assumes zero-alpha with reference to some unobservable zero-beta rate. The latter concept differs from the present empirical approach which, following the lead of Carhart et al. (2014), avoids an equilibrium perspective given our focus on a rather short event window.

It is important to qualify our findings along the arguments of Section 4. With reference to the incomplete model (4.5), our analysis is associated with the ex-post parameter (4.7), for which the inverted tests are conservative. This parameter allows us to analyze the

⁷For comparison with the costliest US hurricanes including Katrina (2005), Sandy (2012), Harvey, and Maria (2017), Michael (2019) and Ian (2022), refer to: (i) the records of the US National Oceanic and Atmospheric Administration [<https://www.noaa.gov/>], in particular the report at <https://www.ncei.noaa.gov/access/billions/dcmi.pdf> on hurricane costs, and (ii) the records of Artemis [<https://www.artemis.bm/>], an established media service on insurance linked investments, in particular “The Artemis Catastrophe Bond Default Directory” at <https://www.artemis.bm/cat-bond-losses>, which compiles Cat bond default data.

bonds’ risk profile for the particular risk scenario which is captured by the full realization of the observed factors. The identifiable *alpha* and *beta* thus depend on the comovements $[X'Z]$ of the observed regressors $[X]$ with the left-out ones $[Z]$, moderated through the “missing *beta*” $[B_Z]$. Our parametrization through (4.7) thus embeds the missing risk factor into the coefficients we interpret, conditioning on the realized returns of included factors. Conservative tests imply that rejections are compelling, which confirms the rejection of zero and negative ex-post *alpha* as identified.

On balance, we do not aim to take a stand on the deep mechanisms which can confound a strict interpretation of *alpha*. Interestingly, Carhart et al. (2014) describes Cat bonds returns in terms of a “continuum” between an equity beta and inefficiency/skill driven *alpha*. Instead, we rely on confidence sets that control statistical coverage from a conservative perspective. Our main finding is that outperformance in the CBMF market post 2011 ensues from the unprecedented character of the considered event, relative to regular catastrophic risk. Moving forward, we emphasize the importance of adapting empirical methods to statistically accommodate the effects of an unobservable factor, on both *alpha* and *beta* components.

8 Conclusion

This paper focuses on finite-sample and simultaneous inference in possibly incomplete and possibly non-Gaussian multivariate regression contexts. The hypotheses considered can affect linear combination of equations, which raises finite-sample complications when disturbances are non-Gaussian. Incomplete settings account for left-out regressors, in which case we show that usual tests, interpreted with reference to a sample-dependent pseudo-true value, remain valid from a conservative perspective. We also derive exact simultaneous confidence bands for a vector of regression coefficients, or for a combination thereof. We illustrate the usefulness of our proposed methodology, to study the risk-return profile of **CBMFs**. Methodologically, our results illustrate the usefulness of finite-sample analysis with standard and incomplete models.

References

- Abdelkhalek, T. and Dufour, J.-M. (1998), ‘Statistical inference for computable general equilibrium models, with application to a model of the moroccan economy’, *The Review of Economics and Statistics* **80**(4), 520–534.
- Akesaka, M. (2019), ‘Change in time preferences: evidence from the great east japan earthquake’, *Journal of Economic Behavior and Organization* **166**, 239–245.
- American Academy of Actuaries (2022), Insurance linked securities and catastrophe bonds, Technical report, American Academy of Actuaries Extreme Events and Property Lines Committee.
- Anderson, T. W. (1984), *An Introduction to Multivariate Statistical Analysis*, 2 edn, Wiley, New York.
- Anderson, T. W. (2003), *An Introduction to Multivariate Statistical Analysis*, 3 edn, Wiley, New York.
- Andrews, D. W. K. and Kwon, S. (2024), ‘Misspecified moment inequality models: Inference and diagnostics’, *The Review of Economic Studies* **91**, 4576.
- Antoine, B., Renault, E. and Proulx, K. (2020), ‘Pseudo-true sdfs in conditional asset pricing models’, *Halbert White Jr. Memorial JFEC Invited Lecture, Journal of Financial Econometrics* **18**, 656714.
- Ardia, D., Barras, L., Gagliardini, P. and Scaillet, O. (2024), ‘Is it alpha or beta? decomposing hedge fund returns when models are misspecified’, *Journal of Financial Economics* **154**, 103805.
- Back, K., Crane, A. D. and Crotty, K. (2018), ‘Skewness consequences of seeking alpha’, *The Review of Financial Studies* **31**, 4720–4761.
- Barone-Adesi, G., Gagliardini, P. and Urga, G. (2004), ‘Testing asset pricing models with coskewness’, *Journal of Business and Economic Statistics* **22**(4), 474–485.

- Beaulieu, M.-C., Dufour, J.-M. and Khalaf, L. (2007), ‘Multivariate tests of mean-variance efficiency with possibly non-gaussian errors’, *Journal of Business and Economic Statistics* **25**(4), 398–410.
- Beaulieu, M.-C., Dufour, J.-M. and Khalaf, L. (2010), ‘Asset-pricing anomalies and spanning: Multivariate and multifactor tests with heavy-tailed distributions’, *Journal of Empirical Finance* **17**(4), 763–782.
- Beaulieu, M.-C., Dufour, J.-M. and Khalaf, L. (2013), ‘Identification-robust estimation and testing of the zero-beta CAPM’, *Review of Economic Studies* **80**, 892–924.
- Beaulieu, M.-C., Dufour, J.-M. and Khalaf, L. (2025), ‘Identification-robust and simultaneous inference in multifactor asset pricing models’, *Journal of Econometrics* **248**, 105915.
- Beaulieu, M.-C., Dufour, J.-M., Khalaf, L. and Melin, O. (2023), ‘Identification-robust beta pricing, spanning, mimicking portfolios, and the benchmark neutrality of catastrophe bonds’, *Journal of Econometrics* **236**, 105464.
- Berk, R., Brown, L., Buja, A., Zhang, K. and Zhao, L. (2013), ‘Valid post-selection inference’, *The Annals of Statistics* **41**(2), 802–837.
- Berndt, E. R. and Savin, N. E. (1977), ‘Conflict among criteria for testing hypotheses in the multivariate linear regression model’, *Econometrica* **45**, 1263–1277.
- Bolduc, D., Khalaf, L. and Yélou, C. (2010), ‘Identification robust confidence set methods for inference on parameter ratios with application to discrete choice models’, *Journal of Econometrics* **157**(2), 317–327.
- Bourdeau-Brien, M. and Kryzanowski, L. (2020), ‘Natural disasters and risk aversion’, *Journal of Economic Behavior and Organization* **117**, 818–835.
- Braun, A. (2016), ‘Pricing in the primary market for CAT bonds: New empirical evidence’, *Journal of Risk and Insurance* **83**(4), 811–847.
- Braun, A., Ben Ammar, S. and Eling, M. (2019), ‘Asset pricing and extreme event risk: Common factors in ILS fund returns’, *Journal of Banking and Finance* **102**, 59–78.

- Braun, A. and Kousky, C. (2021), Catastrophe bonds, Technical report, Risk Management and Decision Processes Center, Wharton, Univeristy of Pennsylvania.
- Carhart, M., Cheah, U.-W., De Santis, G., Farrell, H. and Litterman, R. (2014), ‘Exotic beta revisited’, *Financial Analysts Journal* **70**(5), 24–52.
- Cho, T. (2020), ‘Turning alphas into betas: arbitrage and endogenous risk’, *Journal of Financial Economics* **137**, 550–570.
- Cochrane, J. H. (2007), Efficient markets today, Technical report, Conference on Chicago Economics, Chicago, Illinois.
- Cummins, J. D. and Weiss, M. A. (2009), ‘Convergence of insurance and financial markets: Hybrid and securitized risk-transfer solutions’, *Journal of Risk and Insurance* **76**(3), 493–545.
- Dawid, A. (1981), ‘Some matrix-variate distribution theory: Notational considerations and a Bayesian application’, *Biometrika* **68**, 265–274.
- Dufour, J.-M. (1989), ‘Nonlinear hypotheses, inequality restrictions, and non-nested hypotheses: Exact simultaneous tests in linear regressions’, *Econometrica* **57**(2), 335–355.
- Dufour, J.-M. (1997), ‘Some impossibility theorems in econometrics, with applications to structural and dynamic models’, *Econometrica* **65**, 1365–1389.
- Dufour, J.-M. (2006), ‘Monte carlo tests with nuisance parameters: A general approach to finite-sample inference and nonstandard asymptotics’, *Journal of Econometrics* **133**(2), 443–477.
- Dufour, J.-M. and Khalaf, L. (2002), ‘Simulation based finite and large sample tests in multivariate regressions’, *Journal of Econometrics* **111**(2), 303–322.
- Dufour, J.-M. and Nguyen, V. (2015), Excluded variables and causal analysis in linear regression: interpretation and distributional theory, Technical report, Mc Gill University.

- Dufour, J.-M. and Taamouti, M. (2005), ‘Projection-based statistical inference in linear structural models with possibly weak instruments’, *Econometrica* **73**(4), 1351–1365.
- Dufour, J.-M. and Taamouti, M. (2007), ‘Further results on projection-based inference in IV regressions with weak, collinear or missing instruments’, *Journal of Econometrics* **139**(1), 133–153.
- Fama, E. F. and French, K. R. (1993), ‘Common risk factors in the returns on stocks and bonds’, *Journal of financial economics* **33**(1), 3–56.
- Giglio, S. and Xiu, D. (2021), ‘Asset pricing with omitted factors’, *Journal of Political Economy* **129**(7), 1947–1990.
- Gospodinov, N., Kan, R. and Robotti, C. (2014), ‘Misspecification-robust inference in linear asset-pricing models with irrelevant risk factors’, *The Review of Financial Studies* **27**(7), 2139–2170.
- Götze, M. and Gürtler, M. (2020), ‘Hard markets, hard times: on the inefficiency of the CAT bond market’, *Journal of Corporate Finance* **62**, 101553.
- Gouriéroux, C., Monfort, A. and Renault, E. (1995), Inference in factor models, in G. S. Maddala, P. C. B. Phillips and T. N. Srinivasan, eds, ‘Advances in Econometrics and Quantitative Economics’, Blackwell, Oxford, U.K., chapter 13, pp. 311–353.
- Guggenberger, P., Kleibergen, F. and Mavroeidis, S. (2024), ‘Optimal two-sided tests for instrumental variables regression with heteroskedastic and autocorrelated errors’, *Econometric Theory* **40**, 957–1002.
- Hagendorff, B., Hagendorff, J., Keasey, K. and Gonzalez, A. (2014), ‘The risk implications of insurance securitization: the case of catastrophe bonds’, *Journal of Corporate Finance* **25**, 387–402.
- Hansen, B. E. and Lee, S. (2021), ‘Inference for iterated gmm under misspecification’, *Econometrica* **89**, 14191447.
- Harville, D. A. (1997), *Matrix Algebra From a Statistician’s Perspective*, New York: Springer.

- Herrmann, M. and Hibbeln, M. (2023), ‘Trading and liquidity in the catastrophe bond market’, *Journal of Risk and Insurance* **90**, 283–328.
- Horn, R. A. and Johnson, C. R. (1990), *Matrix Analysis*, Blackwell, New York, U.S.A.
- Jackson, J. E. (2003), *A User’s Guide to Principal Components*, John Wiley & Sons, New York.
- Jolliffe, I. (2004), *Principal Component Analysis*, second edn, Springer, New York.
- Jordà, Ò. (2009), ‘Simultaneous confidence regions for impulse responses’, *The Review of Economics and Statistics* **91**(3), 629–647.
- Jordà, Ò. and Marcellino, M. (2010), ‘Path forecast evaluation’, *Journal of Applied Econometrics* **25**(4), 635–662.
- Kan, R., Robotti, C. and Shanken, J. (2013), ‘Pricing model performance and the two-pass cross-sectional regression methodology’, *The Journal of Finance* **68**(6), 2617–2649.
- Kleibergen, F., Lingwei, K. and Zhan, Z. (2023), ‘Identification robust testing of risk premia in finite samples’, *The Journal of Financial Econometrics* **21**, 263297.
- Kleibergen, F. and Mavroeidis, S. (2023), ‘A test for Kronecker product structure covariance matrix’, *Journal of Econometrics* **233**, 88–112.
- Kleibergen, F. and Zhan, Z. (2018), ‘Identification-robust inference on risk premia of mimicking portfolios of non-traded factors’, *Journal of Financial Econometrics* **16**(2), 155–190.
- Kraus, A. and Litzenberger, R. H. (1976), ‘Skewness preference and the valuation of risk assets’, *Journal of Finance* **31**(4), 1085–1100.
- Lintner, J. (1965), ‘The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets’, *The review of economics and statistics* pp. 13–37.
- Lo, A. W. (2004), ‘The adaptive markets hypothesis’, *Journal of Portfolio Management* **30**, 1529.

- MacKinlay, A. C. (1995), ‘Multifactor models do not explain deviations from the Capital Asset Pricing Model’, *Journal of Financial Economics* **38**, 3–28.
- MacKinlay, A. C. and Pastor, L. (2000), ‘Asset pricing models: Implications for expected returns and portfolio selection’, *Review of Financial Studies* **13**, 883–916.
- Magnus, J. R. and Neudecker, H. (1988), *Matrix Differential Calculus with Applications in Statistics and Econometrics*, New York.
- Montiel Olea, J. L. and Plagborg-Møller, M. (2019), ‘Simultaneous confidence bands: Theory, implementation, and an application to SVARs’, *Journal of Applied Econometrics* **34**, 1–17.
- Moreira, M. and Moreira, H. (2019), ‘Optimal two-sided tests for instrumental variables regression with heteroskedastic and autocorrelated errors’, *Journal of Econometrics* **213**, 398–433.
- Olkin, I. and Tomskey, J. L. (1981), ‘A new class of multivariate tests based on the union-intersection principle’, *The Annals of Statistics* **9**, 792–802.
- Polacek, A. (2018), Catastrophe bonds: a primer and retrospective, Technical report, Chicago Fed Letter No 405.
- Rao, C. R. (1973), *Linear Statistical Inference and its Applications*, 2 edn, Wiley, New York.
- Savin, N. E. (1984), Multiple hypothesis testing, in I. M. E. Griliches, Z., ed., ‘Handbook of Econometrics’, Vol. 2, North-Holland, Amsterdam, pp. 827–879.
- Scheffé, H. (1959), *The analysis of variance*, 1st ed. edn, John Wiley & Sons, New York.
- Schneider, P. Wagner, C. and Zechner, J. (2020), ‘Low-risk anomalies?’, *The Journal of Finance* **75**, 2673–2718.
- Seber, G. A. F. (1984), *Multivariate Observations*, John Wiley & Sons, New York.
- Sharpe, W. F. (1964), ‘Capital asset prices: A theory of market equilibrium under conditions of risk’, *The Journal of Finance* **19**(3), 425–442.

- Zellner, A. (1976), ‘Bayesian and non-bayesian analysis of the regression model with multivariate Student-t error terms’, *Journal of the American Statistical Association* **71**, 400–405.
- Zhao, Y. and Yu, M.-T. (2020), ‘Predicting catastrophe risk: evidence from catastrophe bond markets’, *Journal of Banking and Finance* **121**, 105982.

Appendix

A Proofs

Proof of Theorem 2.1. The proof uses the following matrix algebra result [Harville (1997, section 18.1, p. 416)]: for any $m_1 \times m_2$ matrix A_1 and any $m_2 \times m_1$ matrix A_2 ,

$$\det(I_{m_1} + A_1 A_2) = \det(I_{m_2} + A_2 A_1). \quad (\text{A.1})$$

Given (2.2) and using (A.1), the error ratio $\mathcal{R} := \det[\tilde{S}(\gamma, G)] / \det(\hat{S})$ simplifies to

$$\begin{aligned} \mathcal{R} &= \det\{I_n + G(G' \hat{S} G)^{-1} (C \hat{B} G - D)' [C(X' X)^{-1} C']^{-1} (C \hat{B} G - D) (G' \hat{S} G)^{-1} G' \hat{S}\} \\ &= \det\{I_c + [C(X' X)^{-1} C']^{-1} (C \hat{B} G - D) (G' \hat{S} G)^{-1} G' \hat{S} G (G' \hat{S} G)^{-1} (C \hat{B} G - D)'\} \\ &= \det\{I_c + [C(X' X)^{-1} C']^{-1} (C \hat{B} G - D) (G' \hat{S} G)^{-1} (C \hat{B} G - D)'\} \end{aligned} \quad (\text{A.2})$$

which yields (2.24). ■

Proof of Theorem 3.1. Consider the following decompositions:

$$G' \hat{S} G = G' J W' \mathcal{M}[X] W J' G, \quad (\text{A.3})$$

$$C \hat{B} G - D = C B G - D + C(X' X)^{-1} X' W J' G. \quad (\text{A.4})$$

Thus, under the null hypothesis, $C \hat{B} G - D = C(X' X)^{-1} X' W J' G$ and $S(\gamma, G) - G' \hat{S} G = G' J W' \mathcal{M}_0[X, C] W J' G$, which implies that (2.10) can be rewritten as:

$$\det\{(G' J) W' \mathcal{M}_1 W (J' G)\} = 0 \quad (\text{A.5})$$

where $\mathcal{M}_1 := \mathcal{M}_0(X, C) - \mu \mathcal{M}(X)$, J is invertible, and G has full column rank. This yields equation (3.1). Since J is nonsingular, $\text{rank}(J' G) = \text{rank}(G) = g > 0$. Consider the singular value decomposition of $J' G$ [see Magnus and Neudecker (1988, Chapter 1, Theorem 16)]:

$$J' G = \mathcal{G} \Delta^{1/2} \Xi' \quad (\text{A.6})$$

where Δ is a g -dimensional diagonal matrix which includes the non-zero eigenvalues of $J' G G' J$, $\mathcal{G}$ is a $n \times g$ matrix which includes g orthogonal eigenvectors of $J' G G' J$ (so that $J' G G' J \mathcal{G} = \mathcal{G} \Delta$ and $\mathcal{G}' \mathcal{G} = I_g$), and Ξ is a $n \times g$ matrix which includes g orthogonal eigenvectors of $G' J J' G$ (so that $\Xi' \Xi = I_g$); for $\mathcal{G}$ given, we take $\Xi = G' J \mathcal{G} \Delta^{-1/2}$. On substituting $\mathcal{G} \Delta^{1/2} \Xi'$ in (A.5), we get the equivalent equation

$$\det\{\Xi \Delta^{1/2} \mathcal{G}' W' \mathcal{M}_1 W \mathcal{G} \Delta^{1/2} \Xi'\} = 0 \quad (\text{A.7})$$

hence on taking $\Xi = G'J\mathcal{G}\Delta^{-1/2}$, the following equivalent identities:

$$\begin{aligned} \det\{(G'J\mathcal{G}\Delta^{-1/2})\Delta^{1/2}\mathcal{G}'W'\mathcal{M}_1W\mathcal{G}\Delta^{1/2}(\Delta^{-1/2}\mathcal{G}'J'G)\} &= \det\{(G'J\mathcal{G})\mathcal{G}'W'\mathcal{M}_1W\mathcal{G}(\mathcal{G}'J'G)\} \\ &= \det\{(\mathcal{G}'J'G)(G'J\mathcal{G})\mathcal{G}'W'\mathcal{M}_1W\mathcal{G}\} = \det\{\mathcal{G}'J'GG'J\mathcal{G}\} \det\{\mathcal{G}'W'\mathcal{M}_1W\mathcal{G}\} = 0. \end{aligned} \quad (\text{A.8})$$

Since $\mathcal{G}$ consists of the eigenvectors associated with the non-zero eigenvalues of $J'GG'J$,

$$\mathcal{G}'J'GG'J\mathcal{G} = \mathcal{G}'\mathcal{G}\Delta = \Delta \quad (\text{A.9})$$

and $\det\{\mathcal{G}'J'GG'J\mathcal{G}\} = \det(\Delta) \neq 0$. Thus, equation (A.5) is equivalent to

$$\det\{\mathcal{G}'W'\mathcal{M}_1W\mathcal{G}\} = 0. \quad (\text{A.10})$$

This establishes the characterization (3.2) and completes the proof. ■

Proof of Theorem 3.2. Under the uniformity condition (i), we have: $G = I_n$, and equation (3.1) takes the form

$$\det\{JW'[\mathcal{M}_0(X, C) - \mu\mathcal{M}(X)]WJ'\} = 0 \quad (\text{A.11})$$

which in turn – since $\det(J) \neq 0$ by the invertibility of J – is equivalent to

$$\det\{W'\mathcal{M}_0(X, C)W - \mu W'\mathcal{M}(X)W\} = 0 \quad (\text{A.12})$$

or (3.3) with $\mathcal{W} = W$. Under the separability condition (ii), we have

$$J'G = \begin{bmatrix} J'_1 & J'_2 \\ 0 & J'_3 \end{bmatrix} \begin{bmatrix} I_g \\ 0 \end{bmatrix} = \begin{bmatrix} J'_1 \\ 0 \end{bmatrix} = \begin{bmatrix} I_g \\ 0 \end{bmatrix} J'_1 = GJ'_1 \quad (\text{A.13})$$

so that (3.1) takes the form

$$\det\{J_1G'W'[\mathcal{M}_0(X, C) - \mu\mathcal{M}(X)]WGJ'_1\} = 0 \quad (\text{A.14})$$

which – by the invertibility of J_1 – is equivalent to

$$\det\{G'W'[\mathcal{M}_0(X, C) - \mu\mathcal{M}(X)]WG\} = 0 \quad (\text{A.15})$$

or (3.3) with $\mathcal{W} = WG$. Under the normal mixture assumption (iii), $W = V\mathcal{Z}(n)$ and

$$W\mathcal{G} = V\mathcal{Z}(n)\mathcal{G} \quad (\text{A.16})$$

where, by the orthogonality of the columns of $\mathcal{G}$, all the elements of the matrix $\mathcal{Z}(g) := \mathcal{Z}(n)\mathcal{G}$ are i.i.d. $N(0, 1)$, or equivalently $\text{vec}[\mathcal{Z}(n)\mathcal{G}] \stackrel{i.i.d.}{\sim} N[0, I_{gT}]$. In particular, the distribution of $\mathcal{Z}(n)\mathcal{G}$ does not depend on J . Replacing W by $V\mathcal{Z}(n)$ in equation (3.2):

$$\det\{\mathcal{G}'\mathcal{Z}(n)'V'[\mathcal{M}_0(X, C) - \mu\mathcal{M}(X)]V\mathcal{Z}(n)\mathcal{G}\} = 0, \quad (\text{A.17})$$

$$\det\{\mathcal{Z}(g)'V'[\mathcal{M}_0(X, C) - \mu\mathcal{M}(X)]V\mathcal{Z}(g)\} = 0, \quad (\text{A.18})$$

or (3.3) with $\mathcal{W} = V\mathcal{Z}(g)$. Thus, the distribution of the roots of the determinantal equation (A.18) does not depend on the value of the J matrix. It follows that the same applies to the roots of the equivalent equation (3.3). This completes the proof.

Proof of Proposition 4.1. In the notation of Olkin and Tomsy (1981, Theorem 4.6), set S_1 to $[S(\gamma) - \hat{S}]$ and S_2 to $\hat{S}$. Then applying (2b) with, in their notation, $A = v'$ in which case $k = 1$ yields the result on $\mu_1(\gamma)$; this is because $(AS_1A')(AS_1A')^{-1}$ is a scalar when A is $1 \times p$. Next set S_1 to $G'_i[S(\gamma) - \hat{S}]G_i$ and S_2 to $G'_i\hat{S}G_i$. Applying (2a) with $A = v'_i$ in which case we also have $k = 1$ gives

$$\inf_{v_i} \mathcal{F}_{v_i}(\gamma, G_i) = \mu_{(i)}(\gamma) \quad (\text{A.19})$$

where $\mu_{(i)}(G_i)$ is the largest eigenvalue of $(G'_i[S(\gamma) - \hat{S}]G_i)(G'_i\hat{S}G_i)^{-1}$. Then applying (3b) with $A = G'_i$ yields

$$\mu_i(\gamma) = \frac{c}{T-k} \sup_{G_i} \{\mu_{(i)}(G_i)\} \quad (\text{A.20})$$

which leads to the desired result. ■

Proof of Theorem 4.1. From Proposition 4.1, consider expressions (4.15)-(4.18). Following (4.14)) and based on Rao (1973, Chapter 8): (i) $S(\gamma) - \hat{S}$ and $\hat{S}$ are independent; (ii) $S(\gamma) - \hat{S}$ is a central Wishart distribution with dimension n , degrees-of-freedom $T - k$ and scale JJ' ; and (iii) $\hat{S}$ is a non-central Wishart distribution, with dimension n , degrees-of-freedom c , scale JJ' and non-centrality parameter $\mathcal{M}(X)ZB_Z$. Similarly, for any arbitrary but fixed matrix G_i of rank i , $G'_i[S(\gamma) - \hat{S}]G_i$ is a central Wishart distribution with dimension i , degrees-of-freedom $T - k$ and scale $G'_iJJ'G_i$; and (iii) $\hat{S}$ is a non-central Wishart distribution, with dimension i , degrees-of-freedom c , scale $G'_iJJ'G_i$ and non-centrality parameter $\mathcal{M}(X)ZB_ZG_i$.

This implies that for any *arbitrary but fixed* $n \times 1$ vector a : (i) $a'[S(\gamma) - \hat{S}]a/\sigma_v^2$ follows a central $\chi^2(c)$ distribution with $\sigma_a^2 = a'JJ'a$, (ii) $a'\hat{S}a/\sigma_a^2$ follows a non-central $\chi^2(T - k)$ distribution with non-centrality parameter $v'B_ZZ'\mathcal{M}(X)ZB_Zv/\sigma_v^2$; (iii) $v'[S(\gamma) - \hat{S}]v$ and $v'\hat{S}v/\sigma_v^2$ are independent. Similarly, for any *arbitrary but fixed* $i \times 1$ vector v_i and *arbitrary but fixed* matrix G_i of rank i : (i) $v'_i[G'_i(S - \hat{S})G_i]v_i/\sigma_{v_i}^2(G)$ follows a central $\chi^2(c)$ distribution with $\sigma_{v_i}^2(G_i) = v'_iG'_iJJ'G_iv_i$, (ii) $v'_iG'_i\hat{S}G_iv_i/\sigma_{v_i}^2(G_i)$ follows a non-central $\chi^2(T - k)$ distribution with non-centrality parameter $v'_iG'_iB_ZZ'\mathcal{M}(X)ZB_ZG_iv_i/\sigma_{v_i}^2(G_i)$; (iii) $v'_i[G'_i(S - \hat{S})G_i]v_i$ and $v'_iG'_i\hat{S}G_iv_i$ are independent.

As argued by Dufour and Nguyen (2015), Beaulieu et al. (2023) and Beaulieu et al.

(2025), the null distribution of $\mathcal{F}_v(\gamma)$ can be bounded by the distribution of

$$\bar{\mathcal{F}}_v(C) = \frac{v'U'X(X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}C(X'X)^{-1}X'Uv/c}{v'U'\mathcal{M}(X)Uv/(T-k)} \quad (\text{A.21})$$

in the sense that

$$\Pr[\mathcal{F}_v(\gamma) \geq x] \leq \Pr[\bar{\mathcal{F}}_v(C) \geq x]. \quad (\text{A.22})$$

Similarly, the null distribution of $\mathcal{F}_{v_i}(\gamma, G_i)$ can be bounded by the distribution of

$$\bar{\mathcal{F}}_{v_i}(C, G_i) = \frac{v'_i G'_i U' X (X' X)^{-1} C' [C (X' X)^{-1} C']^{-1} C (X' X)^{-1} X' U G_i v_i / c}{v'_i G'_i U' \mathcal{M}(X) U G_i v_i / (T - k)} \quad (\text{A.23})$$

in the sense that

$$\Pr[\mathcal{F}_{v_i}(\gamma, G_i) \geq x] \leq \Pr[\bar{\mathcal{F}}_{v_i}(C, G_i) \geq x]. \quad (\text{A.24})$$

Now consider $\bar{\mu}_1(C) \geq \dots \geq \bar{\mu}_n(C)$ the roots of the determinantal equation

$$\det\{U'X(X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}C(X'X)^{-1}X'U - \bar{\mu}U'\mathcal{M}(X)U\} = 0 \quad (\text{A.25})$$

which coincide with the roots of

$$\det\{W'X(X'X)^{-1}C'[C(X'X)^{-1}C']^{-1}C(X'X)^{-1}X'W - \bar{\mu}W'\mathcal{M}(X)W\} = 0. \quad (\text{A.26})$$

Applying Olkin and Tomsy (1981) as in Proposition 4.1, we get:

$$\bar{\mu}_1(C) = \frac{c}{T-k} \sup_v \bar{\mathcal{F}}_v(C), \quad (\text{A.27})$$

$$\bar{\mu}_j(C) = \frac{c}{T-k} \sup_{G_i} \inf_{v_i} \bar{\mathcal{F}}_v(C, G_i), \text{ for } j = 2, \dots, n, \quad (\text{A.28})$$

where the optima are taken over the set of $n \times i$ matrices G_i of rank i , and the set of non-zero $i \times 1$ vectors v_i . The function $\lambda_{p,q}(\cdot)$ is component-wise monotonic and order preserving, which yields the dominance results. ■

Proof of Theorem 6.1. The proof is based on the following fundamental matrix algebra result [see Rao (1973, section 1f.1 page 60)]: given a positive definite $n \times n$ matrix A and an $n \times 1$ vector u , we have:

$$\sup_{u \in \mathbb{R}^n} \frac{(v'u)^2}{v'Av} = u'A^{-1}u. \quad (\text{A.29})$$

The generalized Hotelling-type statistics in (2.21) can be written as follows:

$$\Lambda_j(\delta, G) = \frac{[\hat{\beta}_j(G) - \delta]'(G'\hat{S}G)^{-1}[\hat{\beta}_j(G) - \delta]}{s_k[j]'(X'X)^{-1}s_k[j]} = \frac{u'A^{-1}u}{s_k[j]'(X'X)^{-1}s_k[j]} \quad (\text{A.30})$$

where $u := \hat{\beta}_j(G) - \delta$ and $A := G' \hat{S} G$. From (A.29), we then get:

$$u' A^{-1} u = \sup_v \frac{[v'(\hat{\beta}_j(G) - \delta)]^2}{v' G' \hat{S} G v} \quad (\text{A.31})$$

hence

$$\Lambda_j(\delta, G) = \sup_v t_j(v' \delta, G)^2 \quad (\text{A.32})$$

where $t_j(v' \delta, G)$ is defined by (6.5). ■

B Data and Mimicking Portfolios

Prices for seven CBMFs from Bloomberg correspond to the following Tickers: (1) ACATCHA, Falcon Cat Bond Fund Class A CHF; (2) ACATEUA, Falcon Cat Bond Fund Class A EUR; (3) ACATUSA, Falcon Cat Bond Fund Class A USD; (4) ACATCHI, Falcon Cat Bond Fund Class I CHF; (5) LEUPCBC, LGT CH Cat Bond Fund (CHF) A, (6) LEUPCBE, LGT CH Cat Bond Fund (EUR); (7) LEUPCBU, LGT CH Cat Bond Fund (USD) A. Associated returns are expressed in deviation from the one-month Treasury Bill rate, from the Wharton Research Data Services (WRDS). For factors, we use the portfolio of all equities traded on NYSE, AMEX and Nasdaq as the market portfolio. Its value-weighted return (ticker VWRETD), retrieved from the Center for Research in Security Prices (CRSP), is expressed in excess of the risk-free interest rate. The Fama-French benchmarks incorporate the HML and SMB portfolio returns, from the Kenneth French Data Library. For the QCAPM, we use a mimicking portfolio for the squared market return obtained by projecting the latter on a set of excess base asset returns: the momentum factor and six Fama-French portfolios sorted on the size and book-to-market value, from the Kenneth French Data Library, and the US Treasury 10-year government bond yield (ticker USGG10YR) from Bloomberg. Weights are thus obtained following common practice as a function of slope coefficients from the regression of the squared market return on a constant and these base asset returns.