Price Discovery

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Why Study Double Auctions?

- Simple model representing price determination in commodity markets
- Effect of strategic behavior on efficiency
- Impoundment of private information into price
- Equilibrium price verification vs. equilibrium price discovery
- Decentralized identification of the rational expectations price (REE).

Accomplished here:

- Computable model of trading
- Both correlated private (CPV) and interdependent values/costs (CIV)
- Rates of convergence to efficiency and the REE price
- Insight into existence and non-existence of symmetric equilibrium
- Complementarity of *numerical examples* and *theorems*

Earlier Work on Existence and Convergence

- Cripps and Swinkels (2006) in CPV case, Reny and Perry (2006) in CIV case:
  - large numbers of traders
  - no examples
- Results by Satterthwaite and Williams (1989) and Rustichini, Satterthwaite, and Williams (1994)in independent private value (IPV) case

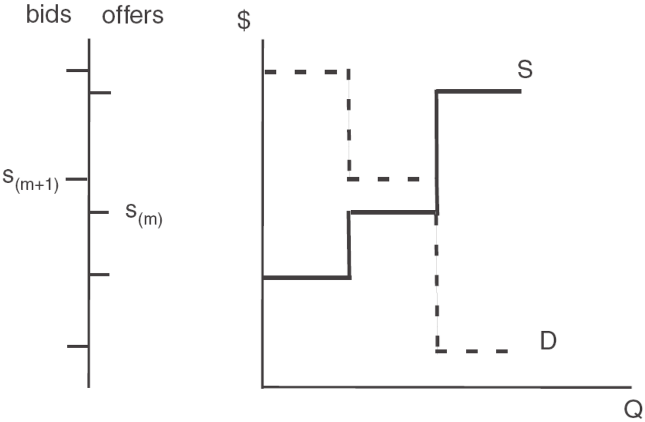
Model

The Buyer's Bid Double Auction (BBDA)

- $m \geq 2$ buyers each of whom wishes to buy one item
- $n \geq 2$ sellers, each of whom wishes to sell one item
- Buyers and sellers simultaneously submit bids/offers
- Bids/offers are ordered in a list:

$$s_{(1)} \leq s_{(2)} \leq \ldots \leq s_{(m)} \leq s_{(m+1)} \leq \ldots \leq s_{(m+n)}$$

- Buyers whose bids are at or above $s_{(m+1)}$ trade with sellers whose offers are below $s_{(m)}$ at the market price of $p = s_{(m+1)}$


bids
offers
$
s(m+1)
s(m)
S
D
Q

Novel Feature: States, Values, Costs, and Signals,

- A state μ is drawn from the *uniform improper prior* on $\mathbb{R}$.
- Buyer i's value is $v_i = \mu + \varepsilon_i$ and seller j's cost is $c_j = \mu + \varepsilon_j$, where ε_i, $\varepsilon_j \backsim G_\varepsilon$.
- A *correlated, private value model (CPV)*: each trader observes a signal σ_i that is his value/cost z_i.
- A *correlated interdependent value model* (CIV): each trader observes a noisy signal $\sigma_i = z_i + \delta_i$ of his value/cost z_i, where $\delta_i \backsim G_\delta$.
- Denote the density of the idiosyncratic component $\varepsilon_i + \delta_i$ by $f_{\varepsilon+\delta}(\cdot)$. Thus $f_{\sigma|\mu}(\sigma|\mu) = f_{\varepsilon+\delta}(\sigma - \mu)$.

The Uniform Improper Prior

- Models complete ignorance about the distribution of values/costs and the likely price ex ante. It is a maximal test of the BBDA institution.
- DeGroot: Usefulness of the diffuse prior
  - forming a prior is costly
  - good information is on the way at the interim stage
  - beliefs conditioned on an observed signal are well-defined

Invariance

- Consider a trader i, a buyer j, and a seller k.
- Conditional on trader i's signal σ_i, the distributions of

$$c_j - \sigma_i,$$
$$v_k - \sigma_i,$$
$$\sigma_j - \sigma_i,$$
$$\sigma_k - \sigma_i,$$

etc., are invariant with the value of $\sigma_i \in \mathbb{R}$.

- What is the implication of this for each trader's strategy?

Offset Strategies

In equilibrium each buyer i uses offset strategy

$$B(\sigma_i) = \sigma_i + \lambda_B$$

and each seller j uses offset strategy

$$S(\sigma_j) = \sigma_j + \lambda_S$$

where $\lambda_B, \lambda_S \in \mathbb{R}$.

First Order Condition: Seller

- The price at which trade occurs is the $(m+1)^{\text{st}}$ order statistic of the ordered bids and asks. The m traders who end up without a unit are the m traders whose bid/ask were the smallest, i.e., $s_{(m+1)}$ clears the market of bids and asks.
- Exclude seller i, the focal seller, from the vector of ordered bids.

- For a small increase Δa_j of his ask seller j's marginal gain is

$$\pi_a^S(a_j|\sigma_j)\,\Delta a_j \approx -\left(a_j - \mathbb{E}[c|_j\sigma_j, a_j < x < a_j + \Delta a_j]\right)\; f^S_{x|\sigma}(a_j|\sigma_j)\,\Delta a_j$$

$$\Rightarrow$$

$$a_j = \mathbb{E}[c_j|\sigma_j, x = a_j]$$

- The seller is a price taker.

First Order Condition: Buyer

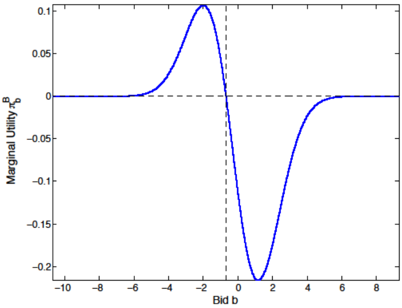
- Exclude buyer i, the focal buyer, from the vector of ordered bids.

- For a small increase Δb_i of her bid, buyer i's marginal gain is

$$\pi_b\left(b_i|\sigma_i\right)\Delta b$$
$$\approx \left(\mathbb{E}[v_i|\sigma_i, x=b_i]-b_i\right) f^B_{x|\sigma}\left(b_i|\sigma_i\right)\Delta b - \Pr\left[x<b_i<y|\sigma_i\right]\Delta b$$
$$\Rightarrow$$
$$b_i=\mathbb{E}[v|\sigma_i, x=b_i]-\frac{\Pr\left[x<b_i<y|\sigma_i\right]}{f^B_{x|\sigma}\left(b_i|\sigma_i\right)}$$

- The linked FOCs define a vector field $\overrightarrow{\mathcal{V}}=\left(\dot{b},\dot{\sigma}_B,\dot{\sigma}_S\right)$ that can be used to graph solution to the FOCs.

Sufficiency of the first order approach verified numerically



- Marginal expected utility for focal buyer ($m = n = 4$, G_ε, G_δ standard normal). The vertical dashed line ($b = -0.7036$) indicates the offset solution to the focal trader's FOC. Seller's marginal utility graph is similar.

Examples of well behaved equilibria

m/n	2	4	16
2	$-1.3404, 0.4124$	$-0.8372, 0.4912$	$0.0361, 0.8546$
4	$-1.2189, 0.1332$	$-0.7036, 0.2172$	$0.1128, 0.6192$
16	$-1.3011, -0.4677$	$-0.8853, -0.3756$	$-0.1754, 0.0614$

Equilibrium offsets λ_B, λ_S for different values of m and n in the case of G_ε, G_δ standard normal

Existence of Solutions to the FOCs

Theorem 1a: Reduction of the FOCs to a single equation

Let $\bar{\lambda} = \bar{\lambda}_B - \bar{\lambda}_S$. Then a necessary and sufficient condition for $\bar{\lambda}_B$ and $\bar{\lambda}_S$ to solve the FOCs is $H\left(\bar{\lambda}\right) = 0$ where

$$\begin{aligned} H\left(\bar{\lambda}\right) =& \frac{\pi_b^B\left(\lambda_B; \lambda_B, \lambda_S\right)}{f_{x|\sigma}^B\left(\lambda_B | \sigma_B = 0\right)} - \frac{\pi_a^S\left(\lambda_S; \lambda_B, \lambda_S\right)}{f_{x|\sigma}^S\left(\lambda_S | \sigma_S = 0\right)} \\ =& -\left(\lambda_B - \lambda_S\right) + \mathbb{E}[v | \sigma_B = 0, x = \lambda_B] - \frac{\Pr\left[x < \lambda_B < y | \sigma = 0\right]}{f_{x|\sigma}^B\left(\lambda_B | \sigma = 0\right)} \\ & - \mathbb{E}[c | \sigma_S = 0, x = \lambda_S]. \end{aligned}$$

$H\left(\bar{\lambda}\right)$depends on λ_B and λ_S only through the difference $\bar{\lambda}$. Invariance allows the signals of the focal buyer and focal seller to be normalized to 0.

First sufficient condition for a solution to FOCs to exist.

Let $\bar{\lambda} = \bar{\lambda}_B - \bar{\lambda}_S = 0$, i.e., $\bar{\lambda}_B = \bar{\lambda}_S$. Then

$$
\begin{aligned}
H\left(\bar{\lambda}\right) &= \frac{\pi_b^B\left(\lambda; \lambda, \lambda\right)}{f_{x|\sigma}^B\left(\lambda | \sigma_B = 0\right)} - \frac{\pi_a^S\left(\lambda; \lambda, \lambda\right)}{f_{x|\sigma}^S\left(\lambda_S | \sigma_S = 0\right)} \\
&= -\left(\lambda_B - \lambda_S\right) + \mathbb{E}[v | \sigma_B = 0, x = \lambda_B] - \frac{\Pr\left[x < \lambda_B < y | \sigma = 0\right]}{f_{x|\sigma}^B\left(\lambda_B | \sigma = 0\right)} \\
&\qquad - \mathbb{E}[c | \sigma_S = 0, x = \lambda_S]
\end{aligned}
$$

Observe that $H\left(0\right) < 0$.

If, for some $\bar{\lambda} < 0, H\left(\bar{\lambda}\right) > 0$, then the intermediate value theorem implies a $\bar{\lambda}^* < 0$ exists such that $H\left(\bar{\lambda}^*\right) = 0$.

Second sufficient condition for a solution to FOCs to exist.

In the case in which the number m of buyers exceeds 1 and G_ε, G_δ satisfy A3, $H\left(\bar{\lambda}\right)$ has a solution in which $\overline{\lambda} < 0$ if

$$\lim_{\overline{\lambda} \to -\infty} \sup \frac{\partial}{\partial \overline{\lambda}} \left(\mathbb{E}[v | \sigma_B = 0, x = \lambda_B] - \mathbb{E}[c | \sigma_S = 0, x = \lambda_S] \right) < 1 - \kappa$$

holds for some $\kappa > 0$. This condition always holds in the CPV case provided $m > 1$ and $G_\varepsilon(\mu)/g_\varepsilon(\mu)$ is increasing.

Why is it hard to prove a general theorem that a solution to the FOCs always exists in the CIV case?

The density $f^S_{x|\mu}(\bar{\lambda}|\mu)$ has the formula

$$f^S_{x|\mu}(\bar{\lambda}|\mu) = (n-1)g_{\varepsilon+\delta}(-\mu)\, K^S_{m,n}(\bar{\lambda},\mu) + mg_{\varepsilon+\delta}(-\bar{\lambda}-\mu)\, L^S_{m,n}(\bar{\lambda},\mu)$$

where

$$\begin{aligned} & K^S_{m,n}(\bar{\lambda},\mu) \\ & = \sum_{\substack{i+k=m-1 \\ 0\le i\le m \\ 0\le k\le n-2}} \frac{m!}{(m-i)!i!}\frac{(n-2)!}{(n-2-k)!k!} \\ & \times \left\{ \begin{array}{c} G_{\varepsilon+\delta}\left(-\left(\bar{\lambda}+\mu\right)\right)^{i} G_{\varepsilon+\delta}(-\mu)^{k} \\ \times G_{\varepsilon+\delta}\left(\bar{\lambda}+\mu\right)^{m-i} G_{\varepsilon+\delta}(\mu)^{n-2-k} \end{array} \right\}. \end{aligned}$$

and $L^S_{m,n}(\bar{\lambda},\mu)$ is similarly ugly.

The price-taking offset

- Suppose buyer's ignore their strategic term, i.e., they act as if they can never affect the price. The price-taking λ_{pt} offset satisfies

$$\lambda_{pt} = \mathbb{E}[z|\sigma = 0, x = \lambda_{pt}].$$

- If buyers ignore the possibility that they can affect price, then both buyers and sellers face the identical winner's curse problem. Therefore λ_{pt} is the price-taking offset for both buyers and sellers.
- Corollary to Theorem 1. The price-taking offset λ_{pt} exists.
- Proof: The RHS depends only on $\bar{\lambda}$, which is 0. λ_{pt} is therefore a straightforward calculation.

Numerical evidence on existence of a solution

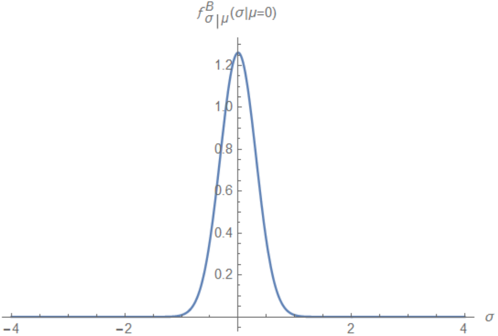
- For $m, n \geq 2$ we have failed to find a case where a solution to the FOCs does not exist. We have experimented with G_ε, G_δ normal, Cauchy, and Laplace.
- The only exception we have found is that a solution does not exist if G_ε, G_δ is Cauchy and $m = n = 1$.

What can go wrong and how does increasing market size fix it

- Show why a solution (λ_B, λ_S) that has unimodal density of the order statistic x is "likely" to be a symmetric equilibrium.
- Present an example with five traders and a bimodal G_ε in which no symmetric equilibrium exists despite existence of a (λ_B, λ_S) solution.
- Explain why, as the number of trader's increases, a symmetric equilibrium necessarily appears.
- For simplicity, these examples are for the CPV case. Any CPV example is easily generalized to the CIV case by adding a small amount of noise to each trader's value. Recall that in the CPV case, seller's always use the offset $\lambda_S = 0$ because they (i) cannot affect the price at which they trade and (ii) a winner's curse does not exist when values/cost are private.

A well behaved CPV example

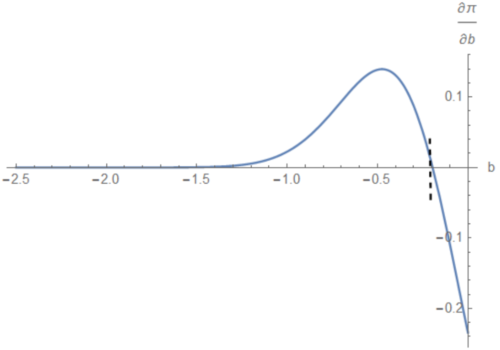
- Three buyers $(m = 3)$ and two sellers $(n = 2)$. G_ε is normal with precision 10. The density of a trader's signal given $\mu = 0$ is:



- The focal buyer's posterior of μ given $\sigma_B = 0$ is the identical density.

Buyer's utility

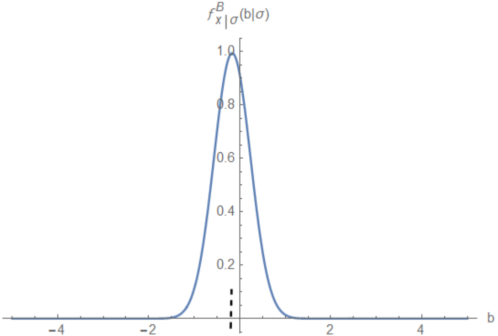
- Given that the focal buyer's signal is $\sigma_B = 0$, the other two buyers are playing $\lambda_B = -0.20$, and both sellers play the zero offset $\lambda_S = 0$, then the focal buyer's utility as a function of his offset is



- The focal buyer's optimal offset is -0.20.

Why doesn't the focal buyer bid lower?

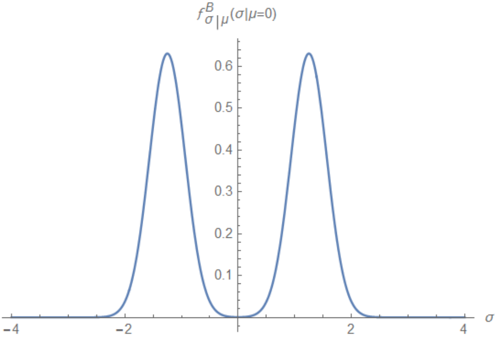
- Condition on the focal buyer's signal $\sigma_B = 0$, other buyers playing $\lambda_B = -0.20$, and all sellers playing a zero offset $\lambda_S = 0$, the density of the critical order statistic x is:



- The focal buyer's optimal offset is -0.20. It optimally trades off a better price against a reduced probability of trade.

A poorly behaved, bimodal, CPV example

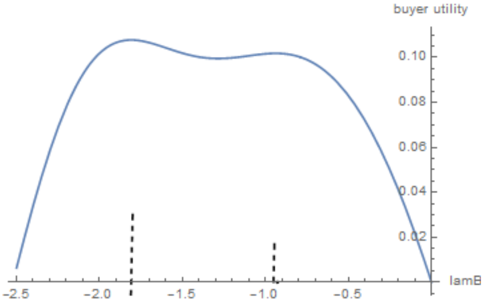
- Three buyers $(m = 3)$ and two sellers $(n = 2)$. G_ε is mixture of two normals $\{\{0.5. - 1.25, 10\}, \{0.5, 1.25, 10\}\}$. The density of a trader's signal given $\mu = 0$ is:



A symmetric solution to the FOCs exists: $\lambda_B = -0.93$ and $\lambda_S = 0$

Buyer's utility

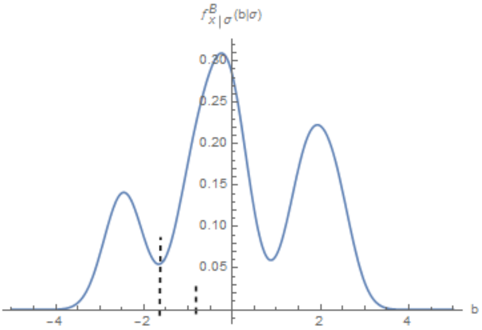
- Conditional on the focal buyer's signal $\sigma_B = 0$, all other buyers playing $\lambda_B = -0.93$, and all sellers playing $\lambda_S = 0$,the focal buyer's utility is:



- The focal buyer deviates with $\lambda'_B = -1.80$.

Why does the focal buyer deviate below the other buyers?

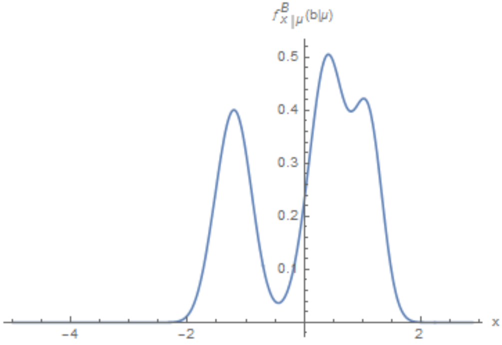
- Conditional on the focal buyer's signal $\sigma_B = 0$, other buyers playing symmetric offsets $\lambda_B = -0.93$, and all sellers playing the zero offset $\lambda_S = 0$, the density of the critical order statistic x is:



- The focal buyer deviates to $\lambda'_B = -1.80$ because of the lump of probability that is to the left of $\lambda'_B = -1.80$

Are symmetric equilibria restored in larger markets?

- What happens to the density of critical order statistic x as the number of buyers and sellers increases?
- Adding one buyer and one seller converts the symmetric solution into an equilibrium.



- The central limit theorem for order statistics kicks in and the distribution of x becomes normal centered on $\mu = 0$.

The Fundamental Convergence Result and a Mathematical Difficulty

Convergence of the strategic term

Theorem. Consider fixed values of m and n, and market sizes $(\eta m, \eta n)$ where $\eta \in \mathbb{N}^+$. Assume that there exists for each $\eta \in \mathbb{N}$ an offset equilibrium $(\lambda_B^\eta, \lambda_S^\eta)$ in the market of size $(\eta m, \eta n)$ such that

$$-L^* < \overline{\lambda}^\eta \equiv \lambda_B^\eta - \lambda_S^\eta < 0$$

for some $L^* > 0$. In each equilibrium in the sequence, a buyer's strategic term is $O(1/\eta)$ in the market with ηm buyers and ηn sellers, i.e., there exists a constant $\mathbb{K}_1(m, n, G_\varepsilon, G_\delta, L^*)$ such that for all values σ_B of a buyer's signal and all $\eta \in \mathbb{N}$,

$$\frac{\Pr[x < \sigma_B + \lambda_B^\eta < y | \sigma_B]}{f_{x|\sigma}^B(\sigma_B + \lambda_B^\eta | \sigma_B)} \leq \frac{\mathbb{K}_1(m, n, G_\varepsilon, G_\delta, L^*)}{\eta}.$$

A Mathematical Difficulty

The mathematical difficulties that stem from a trader's inference problem concerning his value/cost from the market data have prevented us from proving that the $O(1/\eta)$ convergence of the buyer's equilibrium strategic term to zero implies the same rate of convergence of the equilibrium offsets and the price-taking offsets to their common limit. Intuition and examples, however, suggest that the $O(1/\eta)$ rate applies to both. We assume this is so and prove several theorems about the rate at which the BBDA's realized price converges to rational expectation equilibrium price.

Intuition re convergence of equilibrium and price-taking offsets to the limit market's offset

Consider a sequence of offset equilibria $\left((\lambda_B^\eta, \lambda_S^\eta)\right)_{\eta\in\mathbb{N}}$ and price-taking offsets $\left(\lambda_{pt}^\eta\right)_{\eta\in\mathbb{N}}$ for fixed m and n. Apply invariance and normalize the signals σ_B, σ_S of the focal traders to zero. The FOCs are then

$$\lambda_B^\eta = \mathbb{E}[v|\sigma_B = 0, x = \lambda_B^\eta] - O\left(\frac{1}{\eta}\right),$$
$$\lambda_S^\eta = \mathbb{E}[c|\sigma_S = 0, x = \lambda_S^\eta],$$

where $O\left(1/\eta\right)$ represent the focal buyer's strategic term. The FOC for the price-taking offset is

$$\lambda_{pt}^\eta = \mathbb{E}[z|\sigma = 0, x = \lambda_{pt}^\eta].$$

Thus a η becomes large two FOCs converge to the price-taking offset's formula. We therefore conjecture that (i) the respective solutions to the three equations converge to the same limit and (ii) this limit is λ^∞, the equilibrium offset in the limit market. The following numerical claim summarizes our calculations that support this conjecture.

Numerical claim

- Consider a sequence of offset equilibria $\left(\left(\lambda_B^\eta, \lambda_S^\eta\right)\right)_{\eta \in \mathbb{N}^+}$ and price-taking offsets $\left(\lambda_{pt}^\eta\right)_{\eta \in \mathbb{N}^+}$ for fixed m and n. The equilibrium offsets and the price-taking offsets all converge to the equilibrium offset λ^∞ of the limit market at the rate $O(1/\eta)$.
- We have verified this claim for a variety of market sizes when G_ε and G_δ are standard normal, Laplace, or Cauchy,

Some numerical evidence

Panel A

η	λ_B^η	λ_S^η	λ_{pt}^η	$\frac{\Pr[x<\lambda_B^\eta<y\|\sigma_B]}{f_x^B(\lambda_B^\eta\|\sigma_B)}$
2	-1.2190	0.1332	-0.2642	0.7233
4	-0.7431	-0.0788	-0.2837	0.3476
8	-0.5167	-0.1894	-0.2940	0.1679
16	-0.4086	-0.2467	-0.2994	0.0824

Panel B

η	$\|\lambda^\infty - \lambda_B^\eta\|$	$\|\lambda^\infty - \lambda_S^\eta\|$	$\|\lambda^\infty - \lambda_{pt}^\eta\|$
2	0.9144	0.4378	0.0404
4	0.4385	0.2258	0.0209
8	0.2121	0.1152	0.0106
16	0.1040	0.0579	0.0052

Table: Convergence to the limit market in the CIV case ($m = 2$, $n = 1$, G_ε, G_δ standard normal).

Conditional Convergence Results

The classical rational expectations equilibrium price of Radner (1979)

- The *REE function* $P^{\text{REE}} : \mathbb{R} \to \mathbb{R}$ determines the REE price in the limit market for each state μ. It is defined by two properties.
- It is invertible. Let Λ denote the function that recovers the state μ from the REE price, $\Lambda\left(p^{\text{REE}}\right) = \mu$. A REE price p^{REE} is thus *fully revealing* in the sense that a trader who observes p^{REE} can infer the state μ.
- $P^{\text{REE}}(\mu) = p^{\text{REE}}$ clears the limit market in the state μ. Specifically, each trader learns his private signal σ, observes p^{REE}, and calculates his expected value/cost $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}\right), \sigma\right]$. If he is a buyer, he buys one unit if and only if $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}\right), \sigma\right] \geq p^{\text{REE}}$. If he is a seller, he sells his unit if and only if $\mathbb{E}\left[z|\Lambda\left(p^{\text{REE}}\right), \sigma\right] \leq p^{\text{REE}}$. With these choices, demand equals supply at the price p^{REE}.

Convergence of equilibrium price to the rational expectations price: decomposition

- The absolute error in the estimation of p^{REE} with the BBDA's price is bounded above by

$$\left|p_{\mathrm{eq}}^{\eta} - p^{\mathrm{REE}}\right| \leq \left|p_{\mathrm{eq}}^{\eta} - p_{\mathrm{pt}}^{\eta}\right| + \left|p_{\mathrm{pt}}^{\eta} - p^{\mathrm{REE}}\right|$$

  where p_{eq}^{η} is the BBDA's equilibrium price and p_{pt}^{η} is the price-taking price.
- The term $\left|p_{\mathrm{eq}}^{\eta} - p_{\mathrm{pt}}^{\eta}\right|$ is the *strategic error*, which captures the effect of strategic underbidding by buyers upon the estimation of p^{REE} within the BBDA.
- The term $\left|p_{\mathrm{pt}}^{\eta} - p^{\mathrm{REE}}\right|$ is attributable to the fact that a sample of $\eta\,(m+n)$ values/costs does not perfectly reflect the population. together with the error that is attributable to the noise in trader signals. It also reflects the use of the BBDA in determining p_{pt}^{η} as an estimate of p^{REE}. We call this *model error.*

Convergence rate of equilibrium price to the rational expectations equilibrium price

Theorem. Consider distributions G_ε and G_δ that satisfy A1 and tail condition A7, fixed values of m and n, and market size $\eta \in N$. Assume that there exists in each market of size η an offset equilibrium $\left(\lambda_B^\eta, \lambda_S^\eta\right)$ such that $\lambda_B^\eta < \lambda_S^\eta$ for all $\eta \in N$.

- Suppose that both λ_B^η and λ_S^η converge to the offset λ^∞ of the limit market at the rate $O(1/\eta)$. Then the expected total error $E\left[\left|p_{\text{eq}}^\eta - p^{\text{REE}}\right| \mid \mu\right]$ of the equilibrium price as an estimate of the REE price is $\Theta\left(1 \middle/ \sqrt{\eta}\right)$ in each state μ.
- Let λ_{pt}^η denote the price-taking offset of buyers and sellers in the market of size η. Assume that λ_{pt}^η converges to λ^∞ at the rate $O(1/\eta)$. Then the expected model error $E\left[\left|p_{\text{pt}}^\eta - p^{\text{REE}}\right| \mid \mu\right]$ is $\Theta\left(1 \middle/ \sqrt{\eta}\right)$ in each state μ.
- If the hypotheses of both 1. and 2. hold, then the strategic error $\left|p_{\text{eq}}^\eta - p_{\text{pt}}^\eta\right|$ is $O(1/\eta)$ in each state μ and for every sample of signals.

Conclusion

Our analysis show that private information and the strategic behavior that it generates only marginally affect the market's performance relative to price formation, allocative efficiency, and the estimation of the REE price. Except in the smallest of markets, the BBDA discovers price extremely well. Our model is not as general as the models in earlier work,but its restrictiveness allows both formal and computational analysis of finite markets, thus demonstrating that the asymptotic results are meaningful in a finite world.